9 PARABOLOIDS

§ 9.01 THE PARABOLOIDS

The surface represented by the equation

$$ax^2 + by^2 = 2z, \quad ...(1)$$

is not a central conicoid as it does not possess a centre *i.e.*, there is no single point which bisects all chords of the surface passing through itself.

There are two different types of paraboloids represented by the equation (1), according to the coefficients a and b have like or unlike signs. They are known as the elliptic paraboloid and the hyperbolic paraboloid and their standard equations are written in the forms

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z,$$

respectively.

§ 9.02. THE ELLIPTIC PARABOLOID

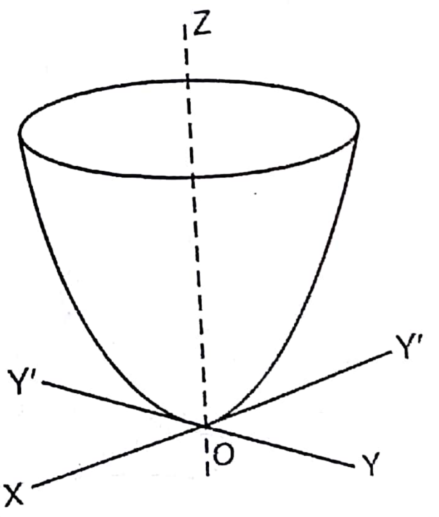
The section of the surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z,$$

by a plane paralled to the xy–plane, is given by the equations

$$z = k, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 2k,$$

which represent an ellipse with its centre on the z–axis, and its axes parallel to x–and y–coordinate axes. The ellipse is real only when k is positive, so that there is no part of the surface below the xy–plane. As k varies from 0 to ∞, the ellipse increases in size from a point- ellipse at O to one of infinite dimens- ions.



The surface is therefore generated by a variable ellipse parallel to the xy–plane and is consequently called the *elliptic paraboloid.*

The section of the surface by a plane parallel to the zx–plane is the parabola given by the equations.

$$y = k \,.\, x^2 = 2a^2\left(z - \frac{k^2}{2b^2}\right),$$

which is real for all values of k, positive or negative. The size of this parabola remains unchanged; only it moves farther away from the origin as k increases numerically. The same is the case with the section parallel to the yz–plane. Thus the paraboloid is also generated by a variable parabola in two different ways.

When $a = b$, the surface becomes a paraboloid of revolution, formed by revolving the parabola

$$y = 0, x^2 = 2a^2 z,$$

about the z–axis.

§ 9.03 THE HYPEROBOLIC PARABOLOID

The section of the surface represented by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z,$$

by a plane parallel to the xy–plane is given by the equations

$$z = k, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 2k,$$

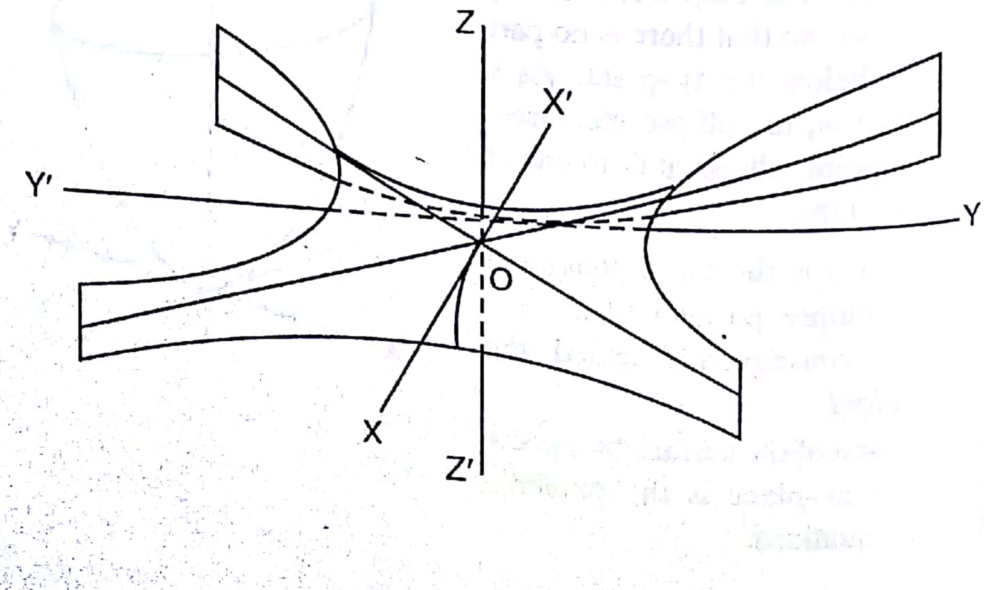
which is a hyperbola, similar to the elliptic section of the previous surface. But this hyperbola is real for all values of k, positive or negative, changing into the conjugate hyperbola as k changes sign. When k is zero, the hyperbola degenerates into the pair of lines

$$z = 0, \frac{x}{a} \pm \frac{y}{b} = 0,$$

which are parallel to the aysmptotes of all hyperbolic sections.

The surface is therefore generated by a variable hyperbola parallel to the xy–plane and it is called the *hyperbolic paraboloid.*

The section of the surface by planes parallel to the zx–or yz– planes is a parabola, similar to the section of the elliptic paraboloid.



§ 9.04. PARABOLOID AS LIMITING FORM OF CENTRAL CONICOID

The elliptic paraboloid can be considered as the limiting form of the ellipsoid or the hyperboloid of two sheets, while the hyperbolic paraboloid can be obtained as the limiting form of the hyperboloid of one sheet. This is seen as follows.

Changing the origin to the point $(0, 0, -c)$, the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ of the ellipsoid becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z-c)^2}{c^2} = 1, \quad i.e., \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{2z}{c}.$$

Now let a, b, c all tend to infinity in such a way that $a^2/c \to \alpha^2$ and $b^2/c \to \beta^2$. Then in the limit it is obvious that the above equation reduces to

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 2z,$$

which is the elliptic paraboloid.

In the same way, the hyperboloid of two sheets, viz. $-x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$ is transformed into $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = -2z$,

which is the elliptic paraboloid with its axis in the negative direction, and the hyperboloid of one sheet, viz. $x^2/a^2 - y^2/b^2 + z^2/c^2 = 1$ is transformed into the hyperbolic paraboloid $\frac{x^2}{\alpha^2} - \frac{y}{\beta^2} = 2z$.

Since the centre of the conicoid, which is the point $(0, 0, c)$ referred to the new origin, moves away to infinity in the limit, we may consider the centre of the paraboloid to be situated at infinity.

§ 9.05 INTERSECTION OF A LINE WITH A PARABOLOID

Let A be the point (α, β, γ), and the equations of a line through A be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

where l, m, n are its direction cosines.

This line intersects the paraboloid $ax^2 + by^2 = 2z$ in points P and Q whose distances from A are given by the roots of the equation

$$r^2(al^2 + bm^2) + 2r(a\alpha l + b\beta m - n) + (a\alpha^2 + b\beta^2 - 2\gamma) = 0. \qquad \text{...(1)}$$

If the line through A is parallel to the z-axis, *i.e.*, if $l = m = 0$, one root of this equation is infinite. Say Q, is at infinity, while the other point P is at a finite distance, and its distance from equation (1) is given by

$$r = \frac{a\alpha^2 + b\beta^2 - 2\gamma}{2n}$$

Such a line is called a *diameter* of the paraboloid and P is the extremity of the diameter.

There is only one point on the surface of the paraboloid, the tangent plane at which is at right angles to the diameter through that point. This point is the *vertex* of the paraboloid and the diameter through the vertex is the *axis* of the paraboloid.

The standard equation therefore represents a paraboloid referred to the vertex as origin, the axis of the paraboloid as the z-axis, and the tangent plane at the vertex and a pair of principal planes as co-ordinate planes.

§ 9.06 TANGENT LINE AND TANGENT PLANE

(Gorakhpur 2007, 08)

If $A(\alpha, \beta, \gamma)$ lies on the surface $(ax^2 + by^2 = 2z)$, we have

$$a\alpha^2 + b\beta^2 - 2\gamma = 0,$$

Which shows that one root of equation (1) of art. 9.05 is zero. The other root is also zero, if in addition $a\alpha l + b\beta m - n = 0$.

When this is true, the line becomes a tangent line to the surface at A.

If we eliminate l, m, n, between this and the equations of the line, we get the locus of all the tangent lines through (α, β, γ), which is given by

$$a\alpha(x-\alpha) + b\beta(y-\beta) - (z-\gamma) = 0,$$

i.e., $$a\alpha x + b\beta y - z = a\alpha^2 + b\beta^2 - \gamma = \gamma,$$

i.e., $$a\alpha x + b\beta y = z + \gamma.$$

This is the equation of the *tangent plane* at (α, β, γ) to the paraboloid.

§ 9.07 CONDITION OF TANGENCY

Let the plane $lx + my + nz = p$ be a tangent plane at (α, β, γ) to the paraboloid $ax^2 + by^2 = 2z$. The equation of tangent plane at (α, β, γ) to the paraboloid is $a\alpha x + b\beta y - z = \gamma$.

Therefore, on comparing, coefficients of x, y, z and constant, we have

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{-1}{n} = \frac{\gamma}{p}.$$

These equations give $\alpha = -\dfrac{l}{an}, \beta = -\dfrac{m}{bn}, \gamma = -\dfrac{p}{n}$.

Since the point (α, β, γ) lies on the paraboloid.

Hence, we have

$$a\left(\frac{l^2}{a^2n^2}\right) + b\left(\frac{m^2}{b^2n^2}\right) = -\frac{2p}{n}, \quad i.e. \quad \frac{l^2}{a} + \frac{m^2}{b} = -2np,$$

which is the required *conditon of tangency*.

Also the point of contact is $\left(-\dfrac{l}{an}, -\dfrac{m}{bn}, -\dfrac{p}{n}\right)$.

Capital. The plane $2n(lx + my + nz) + l^2/a + m^2/b = 0$, is a tangent plane to the paraboloid for all values of l, m, n.

Ex. 1. Show that the plane $2x - 4y - z + 3 = 0$ touches the paraboloid $x^2 - 2y^2 = 3z$., Find also the coordinates of the point of contact.

(Gorakhpur 2005, 10, 12)

Sol. Using the condition $\frac{l^2}{a} + \frac{m^2}{b} + 2np = 0$, it may be verified. Further the equation of the tangent plane at (α, β, γ) to the paraboloid $ax^2 + by^2 = 2z$ is $a\alpha x + b\beta y = z + \gamma$

Comparing the equations of the paraboloids, the coefficients a and b are

$$a = \frac{2}{3}, \; b = -\frac{4}{3}$$

then the equation of tangent plane at (α, β, γ) is $\frac{2}{3}\alpha x - \frac{4}{3}\beta y = z + \gamma$

Comparing it with the given plane, we get

$$\alpha = 3, \; \beta = 3, \; \gamma = -3,$$

the point of contact is $(3, 3, -3)$, the point satisfy the equation of paraboloid.

Ex. 2. Find the condition that the paraloloids $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = \frac{2z}{c_1}$, $\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = \frac{2z}{c_2}$, $\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} = \frac{2z}{c_3}$ may have common tangent plane.

Sol. Let the common tangent plane be $lx + my + nz = p$.

It will be tangent planes to the given paraboloids, if

$$l^2 a_1^2 + m^2 b_1^2 + 2npc_1 = 0$$

$$l^2 a_2^2 + m^2 b_2^2 + 2npc_2 = 0$$

$$l^2 a_3^2 + m^2 b_3^2 + 2npc_3 = 0.$$

On elementaing the unknown l^2, m^2 and $2np$ we get the required conditions as

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix} = 0$$

Ex. 3. Find the locus of the point of intersection of three mutually perpendicular tangent planes to the surface $ax^2 + by^2 = 2z$. *(Gorakhpur 2004, 07)*

Sol. Since the equation of a tangent plane to the given surface in the parametric form is $2n(lx + my + nz) + l^2/a + m^2/b = 0$, we assume that the equations of the three tangent planes which are mutually at right angles, are

$$2n_r(l_r x + m_r y + n_r z) + \frac{l_r^2}{a} + \frac{m_r^2}{b} = 0, r = 1, 2, 3,$$

and that l_r, m_r, n_r are the direction cosines of the normals to these planes. It is obvious that the three lines whose direction cosines are l_1, m_1, n_1, l_2, m_2, n_2; l_3, m_3, n_3 are mutually at right angles, then l_1, l_2, l_3; m_1, m_2, m_3; $n_1 n_2, n_3$ are also direction cosines of three mutually perpendicular st. lines.

Hence $\Sigma n_r l_r = \Sigma m_r n_r = 0$ and $\Sigma l_r^2 = \Sigma m_r^2 = \Sigma n_r^2 = 1$

On adding three equations of the tangent planes, we get

$$2x\Sigma n_r l_r + 2y\,\Sigma m_r n_r + 2z\,\Sigma n_r^2 + \frac{1}{a}\Sigma l_r^2 + \frac{1}{b}\Sigma m_r^2 = 0$$

i.e.

$$2z + 1/a + 1/b = 0$$

Hence the required locus is $2z + 1/a + 1/b = 0$, which is a plane at right angles to the axis of the paraboloid.

EXERCISE

1. Show that the plane $8x - 6y - z = 5$ touches the paraboloid $x^2/2 - y^2/3 = z$, and find the coordinates of the point of contact.

2. Show that the plane $lx + my + nz = p$ touches the paraboloid $ax^2 + by^2 = 2cz$ if $\frac{l^2}{a} + \frac{m^2}{b} + 2\frac{np}{c} = 0$, and the point of contact is then
$$(-lc/an, -mc/bn, -p/n.)$$

3. Prove that the equations to the two tangent planes to the surface $ax^2 + by^2 = 2z$ which pass through the line $u = lx + my + nz - p = 0$, $u' = l'x + m'y + n'z - p' = 0$ is $u^2\left(\frac{l'^2}{a} + \frac{m'^2}{b} + 2n'p'\right)$
$$-2uu'\left(\frac{ll'}{a} + \frac{mm'}{b} + np' + n'p\right) + u'^2\left(\frac{l^2}{a} + \frac{m^2}{b} + 2np\right) = 0.$$

4. Find the condition when plane $lx + my + nz = p$ touch the paraboloid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$ (Purvanchal 2005)

5. Find the locus of the point of intersection of three mutually perpendicular tangent planes to paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ (Gorakhpur 2009)

§ 9.08 THE POLAR PLANE

Let A be the point (α, β, γ), and the equations of a line through A be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r$$

where l, m, n are its direction cosines. If this line intersect the paraboloid $ax^2 + by^2 + 2z$ at points P and Q then AP and AQ are roots of equation (1) of art 9.05. Therefore

$$\therefore \quad AP + AQ = \frac{-2(a\alpha l + b\beta m - n)}{al^2 + bm^2},$$

$$AP.AQ = \frac{a\alpha^2 + b\beta^2 - 2\gamma}{al^2 + bm^2}$$

Let $R(\xi, \eta, \zeta)$ is a harmonic conjugate of A with respect to P and Q and lies on the line APQ. Then

$$AR = \frac{2AP.AQ}{AP + AQ} = -\frac{a\alpha^2 + b\beta^2 - 2\gamma}{a\alpha l + b\beta m - n}$$

and
$$\frac{\xi - \alpha}{l} = \frac{\eta - \beta}{m} = \frac{\zeta - \gamma}{n} = AP$$

Therefore $l\,.AR = \xi - \alpha, m\,.AR = \eta - \beta, n\,.AR = \zeta - \gamma$.

On eliminating l, m, n, we get

$$a\,\alpha\,(\xi - \alpha) = b\beta(\eta - \beta)(\zeta - \gamma) + a\,\alpha^2 + b\beta^2 - 2\gamma = 0,$$

i.e.,
$$a\,\alpha\,\xi + b\,\beta\,\eta - \zeta - \gamma = 0.$$

Hence the locus of R is

$$a\,\alpha\,x + b\,\beta y = z + \gamma,$$

which is the *polar plane* of (α, β, γ) with respect to the paraboloid $ax^2 + by^2 = 2z$.

§ 9.09 THE POLAR LINES

The polar plane of any point $(\alpha + lr,\ \beta + mr, \gamma + nr)$ on the line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n},$$

with respect to the paraboloid $ax^2 + by^2 = 2z$, is

$$ax\,(\alpha + lr) + by\,(\beta + mr) = z + \gamma + nr,$$

i.e.,
$$(a\,\alpha x + b\,\beta y - z - \gamma) + r\,(a\alpha l x + b\beta m y - n) = 0$$

This plane, for all values of r, passes through the line of intersection of the planes $a\,\alpha x + b\,\beta y = z + \gamma$ and $a\alpha l x + b\beta m y - n = 0$, which is the *polar of the given line.*

§ 9.10 ENVELOPING CONE

Let the line
$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \quad ...(1)$$

intersect the paraboloid $ax^2 + by^2 = 2z$ in two coincident points, so that the line is a tangent line from (α, β, γ) to the paraboloid. The condition for this is $(al^2 + bm^2)(a\alpha^2 + b\,\beta^2 - 2\gamma) = (a\,\alpha l + b\,\beta m - n)^2$.

To obtain the locus of the tangent lines, we eliminate l, m, n, from (1). Thus the equation of the *enveloping* cone is

$$\{a\,(x - \alpha)^2 + b(y - \beta)^2\}\,(a\,\alpha^2 + b\,\beta^2 - 2\gamma)$$
$$= \{a\,\alpha\,(x - \alpha) + b\,\beta\,(y - \beta) - (z - \gamma)\}^2,$$

which on simplification becomes

$$(ax^2 + by^2 - 2z)\,(a\,\alpha^2 + b\,\beta^2 - 2\gamma) = (a\,\alpha x + b\,\beta y - z - \gamma)^2, \text{ i.e., } SS_1 = T^2,$$

the symbols having their usual meaning *i.e.,*

$$S = ax^2 + by^2 - 2z,\ S_1 = a\alpha^2 + b\beta^2 - 2\gamma,\ T = a\,\alpha\,x + b\,\beta\,y - z - \gamma$$

§ 9.11 ENVELOPING CYLINDER

As in the preceding Article, the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is a tangent line if

$(al^2+bm^2)(a\alpha^2+b\beta^2-2\gamma)=(a\alpha l+b\beta m-n)^2$.

The locus of the tangent lines parallel to the fixed line $x/l=y/m=z/n$, is therefore given by

$$(al^2+mb^2)(ax^2+by^2-2z)=(alx+bmy-n)^2$$

which is the equation of the *enveloping cylinder.*

§ 9.12 SECTION WITH A GIVEN CENTRE

If (α,β,γ) is the middle point of the chord whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n},$$

the equation (1) of § 9.05 has equal and opposite roots.

Therefore $\quad a\alpha l+b\beta m-n=0.$

Hence the locus of all chords which are bisected at (α,β,γ), is the plane

$$a\alpha(x-\alpha)+b\beta(y-\beta)-(z-\gamma)=0.$$

This can be written as $T=S_1$, where T and S_1 have their usual meaning.

The section of the paraboloid by this plane is a conic such that all chords through (α,β,γ) are bisected at it, *i.e.* (α,β,γ) is the centre of the conic.

§ 9.13 DIAMETRAL PLANE

(Gorakhpur 2004)

Let the chords be parallel to a fixed line OP whose equations are

$$x/l=y/m=z/n.$$

If (α,β,γ) is the mid-point of any one of these chords, we have

$$a\alpha l+b\beta m-n=0.$$

Thus the mid-points of all such chords lie in the plane

$$alx+bmy-n=0.$$

This is the *diametral plane* of OP which is parallel to z–axis.

If OQ is another line given by $x/l'=y/m'=z/n'$, which is parallel to the diametral plane of OP, we have

$$all'+bmm'=0. \quad ...(1)$$

The symmetry of this result shows that OP is parallel to the diametral plane of OQ, and therefore the diametral planes of OP and OQ are *conjugate* to each other.

Ex. 4. Show that the planes $x+3y=3$ and $2x-y=1$ are conjugate diametral planes of the paraboloid $2x^2+3y^2=4z$.

Sol. The equation of any diametral plane with respect to the paraboloid $ax^2+by^2=2z$ is $alx+bmy-n=0$

In this case $a=1, b=\frac{3}{2}$, therefore the equation of diametral plane is

$$2lx+3my-2n=0. \quad ...(1)$$

Let this plane and the given plane $x + 3y = 3$ are same planes, then comparing the coefficients, we have

$$\frac{2l}{1} = \frac{3m}{3} = \frac{+2n}{3}. \qquad \text{or} \qquad \frac{l}{1/2} = \frac{m}{1} = \frac{n}{3/2}.$$

i.e. the direction cosines of the chords which are bisected by the plane $x + 3y = 3$ are proportional to 1, 2, 3. It can be seen that these chords are parallel to the plane $2x - y = 1$. Hence the given planes are conjugate diametral planes.

Ex. 5. Prove that any diametral plane of a paraboloid cuts it in a parabola, and that parallel diametral planes cut it in equal parabolas.

Sol. Let the diametral plane of the paraboloid

$$ax^2 + by^2 = 2z. \qquad \text{...(1)}$$

which bisects chords parallel to the line

$$x/l = y/m = z/n \qquad \text{...(2)}$$

$$a\, lx + b\, my - n = 0. \qquad \text{...(3)}$$

To prove that the section of the paraboloid by the plane (3) is a parabola, it is sufficient to consider the projection of the section on a coordinate plane; for the projection of a conic is a conic of the same species We shall take the projection on the $yz-$ or $zx-$plane but not on the xy–plane in this case, for the plane of the section being parallel to the z–axis, the projection on the xy–plane reduces to a straight line.

Now the projection of the section of the yz–plane is obtained by eliminating x between the equations of the paraboloid and the plane of the section. Therefore the equation of this projection is

$$a(bmy - n)^2/a^2l^2 + by^2 = 2z, \quad i.e. \quad (bmy - n)^2 + al^2\,(by^2 - 2z) = 0$$

i.e. $$b(al^2 + bm^2)y^2 - 2bmny + n^2 = 2al^2\, z.$$

This is obviously a parabola whose latus rectum $2al^2/b(al^2 + bm^2)$ is independent of n.

Hence it also follows that sections by parallal diametral planes are equal parabolas.

EXERCISE

4. The plane $3x + 4y = 1$ is a diametral plane of the paraboloid $5x^2 + 6y^2 = 2z$. Find the equations to the chord through the point (3, 4, 5), which it bisects.

5. Show that the centre of the section of the paraboloid $ax^2 + by^2 = 2z$ by the plane $lx + my + nz = p$, is the point

$$\left(-\frac{l}{an}, -\frac{m}{bn}, \frac{k^2}{n^2}\right), \text{ where } k^2 = \frac{l^2}{a} + \frac{m^2}{b} + np.$$

6. Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the paraboloid $ax^2 + by^2 = 2z$. *(Gorakhpur 2007)*

7. Show that the planes $lx + my + p = 0$ and $l'x + m'y + p' = 0$ are conjugate diametral planes of the paraboloid $ax^2 + by^2 = 2cz$, if $\frac{ll'}{a} + \frac{mm'}{b} = 0$.

(Gorakhpur 2015)

§ 9.14 NORMAL

The equation of the tangent plane at (α, β, γ) to the paraboloid $ax^2 + by^2 = 2z$, is $a\,\alpha x + b\,\beta y = z + \gamma$.

Therefore the *normal* at (α, β, γ) is given by the equations

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-1},$$

where $a\,\alpha, b\,\beta, -1$ are the direction numbers of the normal.

§ 9.15 NORMALS FROM A GIVEN POINT

The normal at any point (x', y', z') of the paraboloid, is

$$\frac{x-x'}{ax'} = \frac{y-y'}{by'} = \frac{z-z'}{-1}.$$

If this passes through a given point (α, β, γ), we have

$$\frac{\alpha-x'}{ax'} = \frac{\beta-y'}{by'} = \frac{\gamma-z'}{-1} = r \text{ (say)},$$

We get
$$x' = \frac{\alpha}{1+ar};\, y' = \frac{\beta}{1+br},\, z' = \gamma + r.$$

But (x', y', z') lies on the paraboloid. Therefore we have

$$\frac{a\,\alpha^2}{(1+ar)^2} + \frac{b\,\beta^2}{(1+br)^2} = 2\,(\gamma + r).$$

This being a fifth degree equation, gives five values of r, from which it follows that there are five points on the paraboloid, the normals at which pass through the given point (α, β, γ). In other words, *five normals in general can be drawn to a paraboloid from a given point.*

§ 9.16 CUBIC CURVE THROUGH THE FEET OF THE NORMALS

If the normal at (x', y', z'), to the paraboloid passes through a given point (α, β, γ), we have as above

$$x' = \frac{\alpha}{1+ar},\, y' = \frac{\beta}{1+br},\, z' = \gamma + r.$$

Thus the parametric equations of the curve passing through the feet of the normals, are

$$x = \frac{\alpha}{1+ar},\, y = \frac{\beta}{1+br},\, z = \gamma + r.$$

The points where this curve meets a given plane, say $ux + vy + wz + d = 0$, are determined by the equation

$$\frac{u\alpha}{1+ar} + \frac{v\beta}{1+br} + w\,(\gamma + r) + d = 0.$$

This is a cubic in r, which gives in general three values of r. Therefore the plane meets the curve in three points, *i.e. the curve through the feet of the normals is a cubic curve.*

The feet of the normals from (α, β, γ) to the paraboloid are the intersections of the paraboloid and this cubic curve.

§ 9.17 CONE THROUGH THE FIVE NORMALS *(Gorakhpur 2006, 15)*

If the line $(x-\alpha)/l = (y-\beta)/m = (z-\gamma)/n$ is the normal at (x', y', z') to the paraboloid, it must be the same as

$$\frac{x-x'}{ax'} = \frac{y-y'}{by'} = \frac{z-z'}{-1}. \text{ Therefore } \frac{l}{ax'} = \frac{m}{by'} = \frac{n}{-1},$$

or, substituting the values of x', y', forms § 9.15

$$\frac{l(1+ar)}{a\alpha} = \frac{m\,(1+br)}{b\,\beta} = \frac{n}{-1}.$$

i.e.,

$$\frac{1/a+r}{\alpha/l} = \frac{1/b+r}{\beta/m} = \frac{-1}{1/n} = \frac{1/a-1/b}{\alpha/l-\beta/m}$$

Hence $\dfrac{\alpha}{l} - \dfrac{\beta}{m} = -\left(\dfrac{1}{a} - \dfrac{1}{b}\right)\dfrac{1}{n}$, and the locus of the normals is the cone

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{b-a}{ab\,(z-\gamma)} = 0.$$

The five normals from (α, β, γ) to the paraboloid are generators of this cone.

Ex. 1. Two perpendicular tangent planes to the paraboloid $x^2/a + y^2/b = 2z$ intersect in a straight line lying in the plane $x = 0$. Show that the line touches the parabola $x = 0, y^2 = (a+b)(2z+a)$.

Sol. Let the straight line in the plane $x = 0$ be given by the equations

$$x = 0, my + nz = p,$$

and let two planes through this line be

$$lx + my + nz = p \quad \text{and} \quad l'x + my + nz = p.$$

These are the tangent planes of the paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b} = 2z$ if

$$al^2 + bm^2 + 2np = 0 \quad \text{and} \quad al'^2 + bm^2 + 2np = 0.$$

Therefore $l^2 = l'^2$, *i.e.*, $l = -l'$.

[Note that $l \neq l'$, for otherwise the two planes would become identical.]

Further, the two planes are at right angles if

$$ll' + m^2 + n^2 = 0.$$

Therefore $l^2 = m^2 + n^2$, *i.e.* $a(m^2 + n^2) + bm^2 + 2np = 0$.

The equations of the line now become

$$x = 0, 2n\,(my + nz) + a(m^2 + n^2) + bm^2 = 0,$$

$$x = 0, (a+b)m^2 + 2mny + (2z + a)\,n^2 = 0.$$

Hence the enevelope of the line is the locus given by the equations

$$x = 0 \quad \text{and} \quad y^2 = (a + b)(2z + a),$$

which respresent a parabola.

Ex. 2. Prove that in general three normals can be drawn from a given po... to the paraboloid of revolution $x^2 + y^2 + 2az$, but if the point lies on the surfa... $27a(x^2 + y^2) + 8(a - z)^3 = 0$, two of the three normals coincide.

Sol. The normal at (x', y', z') to the paraboloid $x^2 + y^2 = 2az$, is

$$\frac{x - x'}{x'} = \frac{y - y'}{y'} = \frac{z - z'}{-a}.$$

This passes through a given point (α, β, γ), if

$$\frac{\alpha - x'}{x'} = \frac{\beta - y'}{y'} = \frac{\gamma - z'}{-a} = \lambda, \text{ say,}$$

so that we have $\alpha - x' = \lambda x'$, or $\alpha = x'(1 + \lambda)$, *i.e.*, $x' = \alpha/(1 + \lambda)$,
and similarly $\quad y' = \beta/(a + \lambda), z' = \gamma + a\lambda.$

Now the point (x', y', z') lies on the paraboloid. Therefoere

$$\alpha^2 + \beta^2 = 2a(\gamma + a\lambda)(1 + \lambda)^2 \quad ...(1)$$

which being a cubic in λ, determines three values of λ, and so three points on th... paraboloid, the normals at which pass through the given point.

The equation (1) has two equal roots if $f(\lambda) = 0, f'(\lambda) = 0$, where

$$f(\lambda) = 2a(\gamma + a\lambda)(1 + \lambda)^2 - \alpha^2 - \beta^2.$$

Thus $\quad 0 = 2a\{a(1 + \lambda^2)^2 + (\gamma + a\lambda) . 2(1 + \lambda)\},$

i.e., $\quad a(1 + \lambda) + 2(\gamma + a\lambda) = 0,$

i.e., $$\frac{1 + \lambda}{-2} = \frac{\gamma + a\lambda}{a} = \frac{a(1 + \lambda) - (\gamma + a\lambda)}{-2a - a} = \frac{\gamma - a}{3a},$$

or, from (1), $$\alpha^2 + \beta^2 = 2a\left(\frac{\gamma - a}{3}\right) . 4\left(\frac{\gamma - a}{3a}\right)^2,$$

i.e., $$27a(\alpha^2 + \beta^2) = 8(\gamma - a)^3.$$

Hence if the point (α, β, γ) lies on the surface

$$27a(x^2 + y^2) = 8(z - a)^3,$$

two of the three normals drawn to the paraboloid coincide.

Ex. 3. Show that the feet of the normals from the point (α, β, γ) to the parabolo... $x^2 + y^2 = 2az$ lies on the sphere

$$x^2 + y^2 + z^2 - y\left(\frac{\alpha^2 + \beta^2}{2\beta}\right) - z(a + \gamma) = 0.$$

Sol. Normal at any point (x', y', z') to the given paraboloid is given by t... equations

$$\frac{x - x'}{x'} = \frac{y - y'}{y'} = \frac{z - z'}{-a}$$

If it passes through (α, β, γ), then

$$\frac{\alpha - x'}{x'} = \frac{\beta - y'}{y'} = \frac{\gamma - z'}{-a} = \lambda \text{ (say)} \Rightarrow x' = \frac{\alpha}{1+\lambda}, y' \frac{\beta}{1+\lambda}, z' = \gamma + a\lambda$$

the feet of the normals are $\left(\frac{\alpha}{1+\lambda}, \frac{\beta}{1+\lambda}, \gamma + a\lambda\right)$

It lies on the paraboloid, therefore

$$\left(\frac{\alpha}{1+\lambda}\right)^2 + \left(\frac{\beta}{1+\lambda}\right)^2 = 2a(\gamma + a\lambda)$$

$$\alpha^2 + \beta^2 = 2a(\gamma + a\lambda)(1+\lambda)^2 \qquad ...(1)$$

The feet of normals lies on the given sphere if

$$\left(\frac{\alpha}{1+\lambda}\right)^2 + \left(\frac{\beta}{1+\lambda}\right)^2 + (\gamma + a\lambda)^2 - \frac{\beta}{1+\lambda}\left(\frac{\alpha^2+\beta^2}{2\beta}\right) - (\gamma + a\lambda)(a+\gamma) = 0$$

$$\frac{\alpha^2+\beta^2}{(1+\lambda)^2} - \frac{\alpha^2+\beta^2}{2(1+\lambda)} + (\gamma + a\lambda)(\gamma + a\lambda - a - \gamma) = 0$$

$$\frac{\alpha^2+\beta^2}{2(1+\lambda)^2}(2-1-\lambda) - a(1-\lambda)(\gamma + a\lambda) = 0$$

$$\frac{(1-\lambda)}{2(1+\lambda)^2}\left[(\alpha^2+\beta^2) - 2a(\gamma + a\lambda)(1+\lambda)^2\right] = 0$$

$$\lambda \neq 1 \qquad \alpha^2 + \beta^2 - 2a(\gamma + a\lambda)(1+\lambda)^2 = 0$$

...ich is same as (1).

EXERCISE

Prove that the cubic cruve through the feet of the normals drawn to a paraboloid from a given point, lies on the cone through the normals themselves.

Prove that the perpendicular from a given point P, to its polar plane with respect to a paraboloid, lies on the cone through the five concurrent normals at P.

Show that the normals from (α, β, γ) to the paraboloid $x^2/a^2 + y^2/b^2 = 2z$, lie on the cone

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0.$$

Prove that five normals can be drawn through a given point (α, β, γ) to the paraboloid $ax^2 + by^2 = 2cz$, and that these normals lie on the cone

(Gorakhpur 2008, 11, 13)

$$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{c(a^{-1} - b^{-1})}{z-\gamma} = 0.$$

The normal at a point $P(\alpha, \beta, \gamma)$ on a paraboloid meets the tangent plane at the vertex in G. Prove that $PG = \gamma^2/p$, where p is the perpendicular from the vertex on the tangent plane at P.

(Gorakhpur 2005, 12)

[**Hint** : $p = -\dfrac{\gamma}{\sqrt{a^2\alpha^2 + b^2\beta^2 + 1}}$. The equation of normal at (α, β, γ) in direction cosine form are $\dfrac{x-\alpha}{\frac{pa\alpha}{\gamma}} = \dfrac{y-\beta}{\frac{pb\beta}{\gamma}} = \dfrac{z-\gamma}{\frac{-p}{\gamma}} = r.$

Since $z = 0$ is the tangent plane at the vertex, so $r = \gamma^2/p$.]

6. Find the equation of the normal to the hyperbolic paraboloid $2x^2 - 3y^2 = 10z$ at the point $(2, 4, -4)$. *(Gorakhpur 2013)*

7. Find the equation of normal at a point (α, β, γ) to the elliphic paraboloid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2z.$ *(Gorakhpur 2014)*