APPLICATIONS OF DE-MOIVRE'S THEOREM

Important Points from the Chapter

1. **De-Moivre's Theorem** Whatever be the value of n, positive, negative or fractional then value or one of the values of $(\cos\theta + i\sin\theta)^n$ is $\cos n\theta + i\sin n\theta$.
2. If p and q are two integers relatively prime to each other, then $(\cos\theta + i\sin\theta)^{p/q}$ has exactly q distinct values.
3. **Expansion of $\cos n\theta$ and $\sin n\theta$, when n is a positive integer**

   We know that

   $$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$
   $$= \cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + {}^nC_3\cos^{n-3}\theta\,(i\sin\theta)^3 + \ldots + (i\sin\theta)^n.$$

   On equating real and imaginary parts on both sides of the above equation, we get

   $$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$$

   and $$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots.$$

Very Short Answer Questions

Q 1. **If $x = \cos\theta + i\sin\theta$ and $\sqrt{1-c^2} = nc - 1$, then show that**

$$1 + c\cos\theta = \frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right).$$ **(2005)**

Sol. We have, $x = \cos\theta + i\sin\theta$, then $\frac{1}{x} = \cos\theta - i\sin\theta$

On adding, we get $x + \frac{1}{x} = 2\cos\theta$ ...(i)

Also, given that, $\sqrt{1-c^2} = nc - 1$

$$1 - c^2 = n^2c^2 + 1 - 2nc$$

$$\Rightarrow 2nc = c^2(1 + n^2)$$

$$\Rightarrow 2n = c(1 + n^2) \quad \ldots(ii)$$

Now, $\text{RHS} = \frac{c}{2n}(1 + nx)\left(1 + \frac{n}{x}\right) = \frac{c}{2n}\left[1 + n^2 + n\left(x + \frac{1}{x}\right)\right]$

$$= \frac{c}{2n}[1 + n^2 + n(2\cos\theta)] \quad \text{[from Eq. (i)]}$$

$$= \frac{c}{2n}\left[\frac{2n}{c} + 2n\cos\theta\right] \quad \text{[from Eq. (ii)]}$$

$$= 1 + c\cos\theta = \text{LHS}$$

Hence proved.

Q 2. Expand $\cos^6\theta$ in a series of cosines of multiplies of θ. **(2003)**

Sol. Let $x = \cos\theta + i\sin\theta$ so that $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Therefore, $x + \frac{1}{x} = 2\cos\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$.

Now, we have $(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6 = {}^6C_0x^6 + {}^6C_1x^5\cdot\frac{1}{x} + {}^6C_2x^4\cdot\frac{1}{x^2}$

$$+ {}^6C_3x^3\cdot\frac{1}{x^3} + {}^6C_4x^2\cdot\frac{1}{x^4} + {}^6C_5x\cdot\frac{1}{x^5} + {}^6C_6\cdot\frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4 + \frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$$\Rightarrow 2^6\cos^6\theta = 2[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$$

$$\Rightarrow \cos^6\theta = \frac{1}{32}[\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10]$$

Q 3. Prove that $32\cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10$. **(2018, 11)**

Sol. Do same as Q. 2.

Q 4. Prove that $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta$. **(2016, 2000)**

Sol. Let $x = \cos\theta + i\sin\theta$ so that $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore$ $x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Therefore, $x - \frac{1}{x} = 2i\sin\theta$ and $x^n - \frac{1}{x^n} = 2i\sin n\theta$.

Now, $(2i\sin\theta)^9 = \left(x - \frac{1}{x}\right)^9$

$$= x^9 - {}^9C_1 x^8 \cdot \frac{1}{x} + {}^9C_2 x^7 \cdot \frac{1}{x^2} - {}^9C_3 x^6 \frac{1}{x^3} + {}^9C_4 x^5 \cdot \frac{1}{x^4} - {}^9C_5 x^4 \cdot \frac{1}{x^5}$$
$$+ {}^9C_6 x^3 \cdot \frac{1}{x^6} - {}^9C_7 x^2 \frac{1}{x^7} + {}^9C_8 x \frac{1}{x^8} - {}^9C_9 \frac{1}{x^9}$$
$$= \left(x^9 - \frac{1}{x^9}\right) - 9\left(x^7 - \frac{1}{x^7}\right) + 36\left(x^5 - \frac{1}{x^5}\right) - 84\left(x^3 - \frac{1}{x^3}\right) + 126\left(x - \frac{1}{x}\right)$$

$\Rightarrow$ $2^9 i\sin^9\theta = 2i\sin 9\theta - 9(2i\sin 7\theta) + 36\,(2i\sin 5\theta)$
$- 84(2i\sin 3\theta) + 126(2i\sin\theta)$

$\Rightarrow$ $2^8\sin^9\theta = \sin 9\theta - 9\sin 7\theta + 36\sin 5\theta - 84\sin 3\theta + 126\sin\theta.$

Hence proved.

***Q* 5.** **Prove that**
$$2^7\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35.$$ **(2014)**

Sol. Let $x = \cos\theta + i\sin\theta$, so that $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore$ $x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Therefore, $x - \frac{1}{x} = 2i\sin\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$.

We have, $(2i\sin\theta)^8 = \left(x - \frac{1}{x}\right)^8$.

$\Rightarrow$ $2^8\sin^8\theta = x^8 - {}^8C_1 x^7 \cdot \frac{1}{x} + {}^8C_2 x^6 \cdot \frac{1}{x^2} - {}^8C_3 x^5 \cdot \frac{1}{x^3} + {}^8C_4 x^4 \cdot \frac{1}{x^4}$
$$- {}^8C_5 x^3 \cdot \frac{1}{x^5} + {}^8C_6 x^2 \cdot \frac{1}{x^6} - {}^8C_7 x \cdot \frac{1}{x^7} + {}^8C_8 \cdot \frac{1}{x^8}$$
$$= \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) - 56\left(x^2 + \frac{1}{x^2}\right) + 70$$
$$= 2\cos 8\theta - 8(2\cos 6\theta) + 28(2\cos 4\theta) - 56\,(2\cos 2\theta) + 70$$

$\Rightarrow$ $2^8\sin^8\theta = 2\,(\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35)$

$\Rightarrow$ $2^7\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$

Hence proved.

***Q* 6.** **Prove that $-2^6\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$.** **(2017, 10)**

Sol. Let $x = \cos\theta + i\sin\theta$, so that $\frac{1}{x} = \cos\theta - i\sin\theta$.

$\therefore$ $x^n = \cos n\theta + i \sin n\theta$ and $\dfrac{1}{x^n} = \cos n\theta - i \sin n\theta$.

Therefore, $x - \dfrac{1}{x} = 2i \sin\theta$ and $x^n - \dfrac{1}{x^n} = 2\, i \sin n\theta$.

We have, $(2i \sin\theta)^7 = \left(x - \dfrac{1}{x}\right)^7$

$$\Rightarrow \quad 2^7 i^7 \sin^7\theta = x^7 - {}^7C_1 x^6 \cdot \frac{1}{x} + {}^7C_2 x^5 \cdot \frac{1}{x^2} - {}^7C_3 x^4 \cdot \frac{1}{x^3} + {}^7C_4 x^3 \cdot \frac{1}{x^4}$$
$$- {}^7C_5 x^2 \cdot \frac{1}{x^5} + {}^7C_6 x \cdot \frac{1}{x^6} - {}^7C_7 \cdot \frac{1}{x^7}$$

$$\Rightarrow \quad -i 2^7 \sin^7\theta = \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right) + 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right)$$

$$\Rightarrow \quad -i 2^7 \sin^7\theta = 2\, i \sin 7\theta - 7(2\, i \sin 5\theta) + 21(2\, i \sin 3\theta) - 35(2\, i \sin\theta)$$

$$\Rightarrow \quad -2^6 \sin^7\theta = \sin 7\theta - 7 \sin 5\theta + 21 \sin 3\theta - 35 \sin\theta.$$

Hence proved.

Short Answer Questions

Q 1. Solve $x^5 = 1$ by De-Moivre's theorem and prove the sum of nth powers of the roots of this equation, n being an integer not divisible by 5, is zero. **(2014)**

Sol. We have, $x^5 = 1 = \cos 2k\pi + i \sin 2k\pi$

$$\Rightarrow \quad x = (\cos 2k\pi + i \sin 2k\pi)^{1/5} = \cos\frac{2k\pi}{5} + i \sin\frac{2k\pi}{5}$$

where, $k = 0, 1, 2, 3, 4$.

Hence, the roots of the above equation are

$$1, \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}, \cos\frac{4\pi}{5} + i \sin\frac{4\pi}{5},$$
$$\cos\frac{6\pi}{5} + i \sin\frac{6\pi}{5} \text{ and } \cos\frac{8\pi}{5} + i \sin\frac{8\pi}{5}.$$

$\therefore$ Sum of nth powers of these roots

$$= 1^n + \alpha^n + \alpha^{2n} + \alpha^{3n} + \alpha^{4n}, \text{ where } \alpha = \cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}.$$

$$= \frac{1 - \alpha^{5n}}{1 - \alpha^n} = \frac{1 - \left(\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}\right)^{5n}}{1 - \left(\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}\right)^n}$$

$$= \frac{1-\cos 2n\pi - i\sin 2n\pi}{1-\cos\frac{2n\pi}{5} - i\sin\frac{2n\pi}{5}} = 0$$

Because numerator is equal to zero, but the denominator is not equal to zero, for n being not a multiple of 5. **Hence proved.**

Q 2. Apply De-Moivre's theorem to solve the equation

$$(1+x)^{2n} + (1-x)^{2n} = 0. \quad \textbf{(2010, 08)}$$

Sol. We have, $(1+x)^{2n} + (1-x)^{2n} = 0$

$$\Rightarrow \quad \left(\frac{1+x}{1-x}\right)^{2n} = -1 = \cos(-\pi) \pm i\sin(-\pi)$$

$$\Rightarrow \quad \left(\frac{1+x}{1-x}\right) = [\cos(2r\pi - \pi) \pm i\sin(2r\pi - \pi)]^{1/2n}$$

$$= \cos\frac{2r\pi - \pi}{2n} \pm i\sin\frac{2r\pi - \pi}{2n}, \; r = 1, 2, 3, \ldots, n$$

Assume $\frac{(2r-1)\pi}{2n} = \theta$, then $\frac{1+x}{1-x} = \frac{\cos\theta \pm i\sin\theta}{1}$

Apply componendo and dividendo, we get

$$\frac{(1+x)-(1-x)}{(1+x)+(1-x)} = \frac{\cos\theta \pm i\sin\theta - 1}{\cos\theta \pm i\sin\theta + 1}$$

$$\Rightarrow \quad x = \frac{-2\sin^2\frac{\theta}{2} \pm 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} \pm 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{2i^2\sin^2\frac{\theta}{2} \pm 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} \pm 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$\Rightarrow \quad x = \pm i\tan\frac{\theta}{2} = \pm i\tan\frac{(2r-1)\pi}{4n}, \text{ where } r = 1, 2, \ldots, n.$$

Q 3. If α, β are roots of $x^2 - 2x + 4 = 0$, prove that

$$\alpha^n + \beta^n = 2^{n+1}\cos\frac{n\pi}{3}. \quad \textbf{(2018)}$$

Sol. Given, $x^2 - 2x + 4 = 0$

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2} = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$x = 1 \pm i\sqrt{3}$$

Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 - i\sqrt{3}$

$$\alpha = 1 + i\sqrt{3} = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$$

$$\beta = 1 - i\sqrt{3} = 2\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]$$

$$\therefore \alpha^n + \beta^n = 2^n\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]^n + 2^n\left[\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right]^n$$

$$= 2^n\left[\left(\cos\frac{n\pi}{3} + i\sin\frac{n\pi}{3}\right) + \left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)\right]$$

$$= 2^n\left[2\cos\frac{n\pi}{3}\right] = 2^{n+1}\cos\frac{n\pi}{3}$$

Hence proved.

Q 4. Use De-Moivre's theorem to solve the equation

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0.$$

(2013, 01)

Sol. We have, $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 = 0$$

$$\Rightarrow \frac{1(1 - x^7)}{1 - x} = 0 \Rightarrow 1 - x^7 = 0, \text{ where } x \neq 1$$

$\Rightarrow$ i.e. $x^7 = 1 = \cos 0 + i\sin 0$

Therefore, $x = (1)^{1/7} = (\cos 0 + i\sin 0)^{1/7}$

$$= (\cos 2k\pi + i\sin 2k\pi)^{1/7}$$

$$= \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}, \text{ where } k = 0, 1, 2, 3, 4, 5, 6$$

$$= \left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)^k$$

i.e. $x = \alpha^k$, where $\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

Thus, the roots are $1, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$ but $x = 1$ is not possible.

Hence, the roots of the given equation are $\alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5, \alpha^6$, where $\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

Q 5. Use De-Moivre's theorem to solve the equation

$$x^4 - x^3 + x^2 - x + 1 = 0.$$

(2016, 11, 07)

Sol. We have, $x^4 - x^3 + x^2 - x + 1 = 0$

$$\Rightarrow \frac{x^5 + 1}{x + 1} = 0, \text{ where } x \neq -1$$

$\Rightarrow x^5 + 1 = 0$ where $x \neq -1$

$$\Rightarrow x^5 = -1 = \cos\pi + i\sin\pi = \cos(2r+1)\pi + i\sin(2r+1)\pi$$

$$\Rightarrow x = \{\cos(2r+1)\pi + i\sin(2r+1\pi\}^{1/5} \quad \text{...(i)}$$

$$= \cos\frac{(2r+1)\pi}{5} + i\sin\frac{(2r+1)\pi}{5}$$

On putting $r = 0, 1, 2, 3, 4$ in Eq. (i), the five roots are given by

$$x_0 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5},$$

$$x_1 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5},$$

$$x_2 = \cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5} = \cos\pi + i\sin\pi = -1,$$

$$x_3 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5} = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5} \quad \left[\because 7\pi = 2\pi - \frac{3\pi}{5}\right]$$

and $$x_4 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5} = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5} \quad \left[\because \frac{9\pi}{5} = 2\pi - \frac{\pi}{5}\right]$$

Since, $x_2 = -1$ so x_2 is not admissible.

Hence, the required roots are x_0, x_1, x_3 and x_4.

Q 6. Use De-Moivre's theorem to solve the equation

$$x^7 + x^5 + x^2 + 1 = 0.$$ **(2004)**

Sol. We have, $x^7 + x^5 + x^2 + 1 = 0$

$\Rightarrow \quad x^5(x^2+1) + 1(x^2+1) = 0 \Rightarrow (x^5+1)(x^2+1) = 0$

$\Rightarrow \quad x^5 + 1 = 0$ or $x^2 + 1 = 0$

When $\quad x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i$ ...(i)

When $\quad x^5 + 1 = 0 \Rightarrow x^5 = -1 \Rightarrow x^5 = \cos\pi + i\sin\pi$

$\Rightarrow \quad x^5 = \cos(2r+1)\pi + i\sin(2r+1)\pi$

$\Rightarrow \quad x = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/5}$

$\Rightarrow \quad x = \cos\dfrac{(2r+1)}{5} + i\sin\dfrac{(2r+1)}{5}\pi$ ...(ii)

where, $r = 0, 1, 2, 3, 4$. Hence, Eqs. (i) and (ii) gives the required roots.

Q 7. Use De-Moivre's theorem to solve the equation

$$x^9 - x^5 + x^4 - 1 = 0.$$ **(2003)**

Sol. We have, $\quad x^9 - x^5 + x^4 - 1 = 0$

$\Rightarrow \quad x^5(x^4-1) + 1(x^4-1) = 0 \Rightarrow (x^5+1)(x^4-1) = 0$

$\Rightarrow \quad (x^5+1)(x^2-1)(x^2+1) = 0$

When $x^2 - 1 = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$ ...(i)

When $x^2 + 1 = 0$, then $x^2 = -1 \Rightarrow x = \pm i$ ...(ii)

When $x^5 + 1 = 0$, then $x^5 = -1 \Rightarrow x^5 = \cos\pi + i\sin\pi$

$\Rightarrow \quad x^5 = \cos(2r+1)\pi + i\sin(2r+1)\pi$

$\Rightarrow \quad x = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/5}$

$\quad = \cos\dfrac{(2r+1)\pi}{5} + i\sin\dfrac{(2r+1)\pi}{5}$ ...(iii)

where, $r = 0, 1, 2, 3, 4$.

Hence, Eqs. (i), (ii) and (iii) gives the required roots.

Q 8. Using De-Moivre's theorem to prove that the roots of the equation $(x-1)^n = x^n$, where n is positive integer, are $\frac{1}{2}\left[1+i\cot\left(\frac{r\pi}{n}\right)\right]$, where $r = 0, 1, 2, 3, \ldots, (n-1)$. **(2008, 2000)**

Or **If n is a positive integer, prove that the roots of the equation $(x-1)^n = x^n$ are $\frac{1}{2}\left[1+i\cot\left(\frac{r\pi}{n}\right)\right]$, where $r = 0, 1, 2, \ldots, (n-1)$.** **(2012)**

Or **Use De-Moivre's theorem to solve the equation $(x-1)^n = x^n$.** **(2017)**

Sol. We have, $(x-1)^n = x^n \Rightarrow \left(\frac{x-1}{x}\right)^n = 1 = \cos 0 + i\sin 0$

$$\Rightarrow \quad \frac{x-1}{x} = (\cos 2r\pi + i\sin 2r\pi)^{1/n} = \cos\frac{2r\pi}{n} + i\sin\frac{2r\pi}{n},$$

where $r = 0, 1, \ldots, n-1$.

Therefore, $1 - \frac{1}{x} = \frac{\cos 2r\pi}{n} + i\sin\frac{2r\pi}{n}$.

$$\Rightarrow \quad x = \frac{1}{1-\cos\frac{2r\pi}{n} - i\sin\frac{2r\pi}{n}}$$

$$= \frac{1}{1-1+2\sin^2\frac{r\pi}{n} - 2i\sin\frac{r\pi}{n}\cos\frac{r\pi}{n}}$$

$$= \frac{1}{-2i\sin\frac{r\pi}{n}\left(\cos\frac{r\pi}{n} + i\sin\frac{r\pi}{n}\right)}$$

$$= \frac{1}{-2i\sin\frac{r\pi}{n}}\left(\cos\frac{r\pi}{n} - i\sin\frac{r\pi}{n}\right) = \frac{1}{2}\left[i\cot\frac{r\pi}{n} + 1\right]$$

Thus, $x = \frac{1}{2}\left[1 + i\cot\left(\frac{r\pi}{n}\right)\right]$ **Hence proved.**

Q 9. Apply De-Moivre's theorem to solve the equation **(2009)**

$$x^5 - x^4 + x^3 - x^2 + x - 1 = 0.$$

Sol. We have, $x^5 - x^4 + x^3 - x^2 + x - 1 = 0$

$$\Rightarrow \quad 1 - x + x^2 - x^3 + x^4 - x^5 = 0 \qquad \left[\because S_n = \frac{a(1-r^n)}{1-r}, r < 1\right]$$

$$\Rightarrow \quad \frac{1[1-(-x)^6]}{1-(-x)} = 0$$

$$\Rightarrow \quad \frac{1-x^6}{1+x} = 0 \Rightarrow 1 - x^6 = 0, \text{ where } x \neq -1$$

$$x^6 = 1 = \cos 0 + i \sin 0$$

$$\Rightarrow \quad x^6 = \cos 2k\pi + i \sin 2k\pi$$

$$\Rightarrow \quad x = (\cos 2k\pi + i \sin 2k\pi)^{1/6} = \cos \frac{2k\pi}{6} + i \sin \frac{2k\pi}{6},$$

where $k = 0, 1, 2, 3, 4, 5$.

$$\Rightarrow \quad x = \cos \frac{k\pi}{3} + i \sin \frac{k\pi}{3} \quad \text{...(i)}$$

On putting $k = 0, 1, 2, 3, 4, 5$ in Eq. (i), we have

$$x_0 = \cos 0 + i \sin 0 = 1,$$

$$x_1 = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = \frac{1+\sqrt{3}i}{2},$$

$$x_2 = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = \frac{-1+\sqrt{3}i}{2},$$

$$x_3 = \cos \pi + i \sin \pi = -1,$$

$$x_4 = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = \frac{-1-\sqrt{3}i}{2},$$

$$x_5 = \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3} = \frac{1-\sqrt{3}i}{2}.$$

But $x = -1$ is not possible.

Hence, the roots of the given equation are $1, \dfrac{\pm 1 \pm \sqrt{3}\,i}{2}$.

Q 10. If p and q are the integers relatively prime to each other, then prove that $(\cos\theta + i \sin\theta)^{p/q}$ has exactly q distinct values which can be arranged in a GP. **(2015)**

Sol. By De-Moivre's theorem, one of the values of

$$(\cos\theta + i \sin\theta)^{p/q} \text{ is } \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}.$$

We know that cosine and sine of an angle have the same values when the angle is increased by any integeral multiple of 2π.

Hence, $(\cos\theta + i \sin\theta)^{p/q} = [\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)]^{p/q}$

$$\Rightarrow \quad (\cos\theta + i \sin\theta)^{p/q} = \cos \frac{p}{q}(\theta + 2k\pi) + i \sin \frac{p}{q}(\theta + 2k\pi),$$

where $k = 0, 1, 2, \ldots$

The q different values of $(\cos\theta + i \sin\theta)^{p/q}$ are for $k = 0, 1, 2, \ldots, q-1$ in the right of the above equality, such as

$$x_0 = \cos \frac{p\theta}{q} + i \sin \frac{p\theta}{q}$$

$$x_1 = \cos\left(\frac{p\theta}{q} + \frac{2\pi p}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2\pi p}{q}\right)$$

....

....

$$x_{q-1} = \cos\left(\frac{p\theta}{q} + \frac{2(q-1)\,\pi p}{q}\right) + i\sin\left(\frac{p\theta}{q} + \frac{2(q-1)\pi p}{q}\right). \quad \text{... (i)}$$

No two of the q values given by Eq. (i) are the same. Since, the angles involved here is differ by less than 2π. The difference of the greatest and the least angles is

$$\left(\frac{p\theta}{q} + \frac{2\pi(q-1)p}{q}\right) - \frac{p\theta}{q} = \frac{(q-1)2\pi p}{q},$$

which is less than a multiple of 2π.

Since, $q-1 < q$. Hence, the q values given by Eq. (i) are distinct.

By giving to k the values $q, q+1, q+2, \ldots$, we obtain no new values of the given expression and only the old values are repeated.

Thus, we see that $(\cos\theta + i\sin\theta)^{p/q}$, where p and q are integers prime to each other, has q and only q distinct values and these are obtained on successively putting $k = 0, 1, 2, \ldots, (q-1)$ in the expression

$$\cos\left\{\frac{p}{q}(\theta + 2k\pi)\right\} + i\sin\left\{\frac{p}{q}(\theta + 2k\pi)\right\}.$$

Now, the expression $\cos\frac{p}{q}(\theta + 2k\pi) + i\sin\frac{p}{q}(\theta + 2k\pi)$

$$= \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos 2k\pi\frac{p}{q} + i\sin 2k\pi\frac{p}{q}\right)$$

$$= \left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right)\left(\cos 2\pi\frac{p}{q} + i\sin 2\pi\frac{p}{q}\right)^k$$

From this, we obtain by putting $k = 0, 1, 2, \ldots, q-1$, the q distinct values as $\alpha, \alpha\beta, \alpha\beta^2, \alpha\beta^3, \ldots, \alpha\beta^{q-1}, \ldots$

where, $\alpha = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$, $\beta = \cos\frac{2\pi p}{q} + i\sin\frac{2\pi p}{q}$.

Hence the q distinct values of $(\cos\theta + i\sin\theta)^{p/q}$ can be arranged in a GP.

Hence proved.

Q 11. If $\theta_1, \theta_2, \theta_3$ be the values of θ, which satisfy the equation $\tan 2\theta = \lambda\tan(\theta + \alpha)$ and if no two of these values differ by a multiple of α, show that $\theta_1 + \theta_2 + \theta_3 + \alpha$ is a multiple of π.

(2008)

Sol. We have, $\tan 2\theta = \lambda\tan(\theta + \alpha)$

$$\Rightarrow \quad \frac{2\tan\theta}{1-\tan^2\theta} = \lambda\cdot\frac{\tan\theta + \tan\alpha}{1-\tan\theta\tan\alpha}$$

$$\Rightarrow \quad \lambda \tan^3\theta + (\lambda - 2)\tan\alpha \tan^2\theta + (2-\lambda)\tan\theta - \lambda\tan\alpha = 0$$

which is a cubic equation in $\tan\theta$ and so it has three roots $\tan\theta_1$, $\tan\theta_2$ and $\tan\theta_3$.

$$\therefore \quad S_1 = \tan\theta_1 + \tan\theta_2 + \tan\theta_3 = \frac{(2-\lambda)\tan\alpha}{\lambda}$$

$$S_2 = \tan\theta_1\tan\theta_2 + \tan\theta_2\tan\theta_3 + \tan\theta_3\tan\theta_1 = \frac{2-\lambda}{\lambda}$$

and $$S_3 = \tan\theta_1\tan\theta_2\tan\theta_3 = \frac{\lambda\tan\alpha}{\lambda} = \tan\alpha.$$

Now, we have $$\tan(\theta_1+\theta_2+\theta_3) = \frac{S_1 - S_3}{1 - S_2} = \frac{\frac{(2-\lambda)\tan\alpha}{\lambda} - \tan\alpha}{1 - \frac{2-\lambda}{\lambda}}$$

$$= \frac{-(\lambda-1)\tan\alpha}{(\lambda-1)} = -\tan\alpha = \tan(-\alpha)$$

Hence, the general value is given by

$$\theta_1 + \theta_2 + \theta_3 = n\pi - \alpha$$

i.e. $\theta_1 + \theta_2 + \theta_3 + \alpha = n\pi$ = a multiple of π. **Hence proved.**

Q 12. **If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + a = 0$, prove that $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$ radians except when $b = 1$.** **(2014, 09)**

Sol. Since, α, β and γ are the roots of the equation $x^3 + ax^2 + bx + a = 0$.

Therefore, $\Sigma\alpha = \alpha + \beta + \gamma = S_1 = -a$,

$$\Sigma\alpha\beta = \alpha\beta + \alpha\gamma + \beta\gamma = S_2 = b \text{ and } \alpha\beta\gamma = S_3 = -a.$$

Let us suppose that $\tan^{-1}\alpha = \theta_1$, $\tan^{-1}\beta = \theta_2$, $\tan^{-1}\gamma = \theta_3$

Then, $$\tan(\theta_1+\theta_2+\theta_3) = \frac{S_1 - S_3}{1 - S_2} = \frac{-a-(-a)}{1-b} = 0, \text{ except } b = 1.$$

$$\Rightarrow \quad \theta_1 + \theta_2 + \theta_3 = n\pi$$

$\Rightarrow \quad \tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi$, except $b = 1$. **Hence proved.**

Q 13. **Prove that the equation $\sin 3\theta = a\sin\theta + b\cos\theta + c$ has six roots and that the sum of the six values of θ which satisfy it, is equal to an odd multiple of π radian.** **(2011)**

Sol. The given equation is $\sin 3\theta = a\sin\theta + b\cos\theta + c$

i.e. $$3\sin\theta - 4\sin^3\theta = a\sin\theta + b\cos\theta + c$$

On putting $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$, where $t = \tan\frac{\theta}{2}$

We have, $$3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right)^3 = a\left(\frac{2t}{1+t^2}\right) + b\left(\frac{1-t^2}{1+t^2}\right) + c$$

$$(b-c)t^6 - 2(a-3)t^5 + (b-3c)t^4 - 4(a+5)t^3 - (b+3c)t^2$$

$\Rightarrow$ $$-2(a-3)t - (b+c) = 0$$

The above equation is of the six degree in t, so it will have six roots say t_1, t_2, t_3, t_4, t_5 and t_6.

$\therefore$ $$S_1 = \Sigma t_1 = \frac{2(a-3)}{b-c}, \quad S_2 = \Sigma t_1 t_2 = \frac{(b-3c)}{b-c}$$

$$S_3 = \Sigma t_1 t_2 t_3 = \frac{4(a+5)}{b-c}, \quad S_4 = \Sigma t_1 t_2 t_3 t_4 = \frac{-(b+3c)}{b-c}$$

and $$S_5 = \Sigma t_1 t_2 t_3 t_4 t_5 = \frac{2(a-3)}{b-c}, \quad S_6 = \Sigma\, t_1 t_2 t_3 t_4 t_5 t_6 = \frac{-(b+c)}{b-c}$$

Let $\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$ and θ_6 be the values of θ corresponding to the six values of t, then

$$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}+\frac{\theta_5}{2}+\frac{\theta_6}{2}\right) = \frac{S_1 - S_3 + S_5}{1 - S_2 + S_4 - S_6} = \frac{S_1 - S_3 + S_5}{0} = \infty$$

$\Rightarrow$ $$\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}+\frac{\theta_5}{2}+\frac{\theta_6}{2} = n\pi + \frac{\pi}{2}$$

$\Rightarrow$ $$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \theta_6 = 2n\pi + \pi$$

$$= (2n+1)\pi = \text{an odd multiple of } \pi$$

Hence proved.

Q 14. Find the fifth roots of unity. Prove that these roots are in G.P. and the sum of their nth powers always vanishes unless n is a multiple of 5, in which case their sum is 5, n being an integer.

(2014)

Sol. Let $x = (1)^{1/5} = (\cos 2r\pi + i \sin 2r\pi)^{1/5}$, where $r = 0, 1, 2, 3, 4$

$\Rightarrow$ $$x = \cos\frac{2r\pi}{5} + i \sin\frac{2r\pi}{5} \quad \text{...(i)}$$

On putting the values of r in Eq. (i), we have

$$x_0 = \cos 0 + i \sin 0 = 1$$

$$x_1 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5} = \alpha \text{ (say)}$$

$$x_2 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = \alpha^2 \text{ (say)}$$

$$x_3 = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = \alpha^3 \text{ (say)}$$

and $$x_4 = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \alpha^4 \text{ (say)}$$

where x_k denotes the value of x, obtained by putting $k = 0, 1, 2, 3, 4$ in Eq. (i).

Obviously x_0, x_1, x_2, x_3, x_4 are in geometric progression.

Case I When n is not a multiple of 5.

Let $$S = (x_0)^n + (x_1)^n + (x_2)^n + (x_3)^n + (x_4)^n$$

$$= 1 + \alpha^n + \alpha^{2n} + \alpha^{3n} + \alpha^{4n} = \frac{1(1-\alpha^{n5})}{1-\alpha^n} = \frac{1-\alpha^{5n}}{1-\alpha^n}$$

$$= \frac{0}{1-\alpha^n} \qquad [\because \alpha^{5n} = 1]$$

$$= 0 \qquad [\because \alpha^n \neq 1, n \text{ is not a multiple of } 5]$$

Case **II** When n is a multiple of 5, then

$$\alpha^n = \alpha^{5m} = \cos 2m\pi + i \sin 2m\pi = 1$$

Similarly, $\alpha^{2n} = \alpha^{3n} = \alpha^{4n} = 1.$

$\therefore$ $S = 1+1+1+1+1 = 5$

Hence, when n is a multiple of 5, then their sum is 5. **Hence proved.**

***Q* 15. Prove that the equation $ah \sec\theta - bk \operatorname{cosec}\theta = a^2 - b^2$ has four roots and that the sum of the four values of 'θ' which satisfy it is equal to an odd multiple of π radians.** **(2016)**

Sol. We have, $ah \sec\theta - bk \operatorname{cosec}\theta = a^2 - b^2$

i.e. $$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2. \qquad \text{...(i)}$$

Now, $$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2} = \frac{1-t^2}{1+t^2}$$

and $$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2} = \frac{2t}{1+t^2}, \text{ where } t = \tan\theta/2.$$

On putting these values of $\cos\theta$ and $\sin\theta$ in Eq. (i), we get

$$ah\left(\frac{1+t^2}{1-t^2}\right) - bk\left(\frac{1+t^2}{2t}\right) = a^2 - b^2$$

$\Rightarrow$ $$bk\,t^4 + (2ah + 2a^2 - 2b^2)t^3 + (2ah - 2a^2 + 2b^2)t - bk = 0,$$

which is an equation of degree four in t and so it has four roots t_1, t_2, t_3, t_4 (say).

We have, $$S_1 = \Sigma t_1 = -\frac{2ah + 2a^2 - 2b^2}{bk}, \; S_2 = \Sigma t_1 t_2 = 0$$

$$S_3 = \Sigma t_1 t_2 t_3 = -\frac{2ah - 2a^2 + 2b^2}{bk}, \; S_4 = \Sigma t_1 t_2 t_3 t_4 = -1$$

Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the values of θ corresponding to the values t_1, t_2, t_3, t_4, respectively.

Then, $$\tan\left(\frac{\theta_1}{2} + \frac{\theta_2}{2} + \frac{\theta_3}{2} + \frac{\theta_4}{2}\right) = \frac{S_1 - S_3}{1 - S_2 + S_4} = \infty = \tan\frac{\pi}{2}.$$

Therefore, the general value is given by

$$\frac{\theta_1 + \theta_2 + \theta_3 + \theta_4}{2} = n\pi + \frac{\pi}{2} = (2n+1)\frac{\pi}{2}$$

i.e. $\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n+1)\pi$ = an odd multiple of π radians.

Hence proved.

Q 16. Prove that
$$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta.$$
(2001)

Sol. By De-Moivre's theorem, we have
$$(\cos\theta + i\sin\theta)^7 = \cos 7\theta + i\sin 7\theta$$
$$\Rightarrow \cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$$
$$= {}^7C_0\cos^7\theta + {}^7C_1\cos^6\theta(i\sin\theta) + {}^7C_2\cos^5\theta(i\sin\theta)^2 + {}^7C_3\cos^4\theta(i\sin\theta)^3 + {}^7C_4\cos^3\theta(i\sin\theta)^4 + {}^7C_5\cos^2\theta(i\sin\theta)^5 + {}^7C_6\cos\theta(i\sin\theta)^6 + {}^7C_7(i\sin\theta)^7$$
$$\Rightarrow \cos 7\theta + i\sin 7\theta = \cos^7\theta + i\,{}^7C_1\cos^6\theta\sin\theta - {}^7C_2\cos^5\theta\sin^2\theta - i\,{}^7C_3\cos^4\theta\sin^3\theta + {}^7C_4\cos^3\theta\sin^4\theta + i\,{}^7C_5\cos^2\theta\sin^5\theta - {}^7C_6\cos\theta\sin^6\theta - i\,{}^7C_7\sin^7\theta$$
$$\Rightarrow \cos 7\theta + i\sin 7\theta = [\cos^7\theta - {}^7C_2\cos^5\theta\sin^2\theta + {}^7C_4\cos^3\theta\sin^4\theta - {}^7C_6\cos\theta\sin^6\theta] + i[{}^7C_1\cos^6\theta\sin\theta - {}^7C_3\cos^4\theta\sin^3\theta + {}^7C_5\cos^2\theta\sin^5\theta - \sin^7\theta]$$
On comparing the imaginary part, we get
$$\sin 7\theta = {}^7C_1\cos^6\theta\sin\theta - {}^7C_3\cos^4\theta\sin^3\theta + {}^7C_5\cos^2\theta\sin^5\theta - \sin^7\theta$$
$$= 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$$
Hence proved.

Q 17. Prove that
$$\tan^2\frac{\pi}{11} + \tan^2\frac{2\pi}{11} + \tan^2\frac{3\pi}{11} + \tan^2\frac{4\pi}{11} + \tan^2\frac{5\pi}{11} = 55.$$
(2015)

Sol. Let $\theta = \frac{n\pi}{11}$, where $n = 1, 2, 3, 4, 5$.

Then, $11\theta = n\pi$ and $\tan 11\theta = \tan n\pi = 0$.

Therefore, ${}^{11}C_1\tan\theta - {}^{11}C_3\tan^3\theta + {}^{11}C_5\tan^5\theta - {}^{11}C_7\tan^7\theta + {}^{11}C_9\tan^9\theta - \tan^{11}\theta = 0$
$$\Rightarrow 11\tan\theta - 165\tan^3\theta + 462\tan^5\theta - 330\tan^7\theta + 55\tan^9\theta - \tan^{11}\theta = 0$$
$$\Rightarrow \tan\theta(\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11) = 0$$
Therefore, either $\tan\theta = 0$, i.e. $\theta = n\pi$ which is not possible.

or $\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11 = 0$

Put $\tan^2\theta = x$, then it becomes $x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 = 0$, which is the required equation.

The above equation in x is of degree 5 and therefore it has five roots x_1, x_2, x_3, x_4, x_5, say as $\tan^2\frac{\pi}{11}, \tan^2\frac{2\pi}{11}, \tan^2\frac{3\pi}{11}, \tan^2\frac{4\pi}{11}, \tan^2\frac{5\pi}{11}$.

Now, $S_1 = \Sigma x_1 = \tan^2\frac{\pi}{11} + \tan^2\frac{2\pi}{11} + \tan^2\frac{3\pi}{11} + \tan^2\frac{4\pi}{11} + \tan^2\frac{5\pi}{11} = 55$.

Hence proved.

Q 18. Prove that $\frac{1}{6}\sin^3\theta = \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7!} - \ldots$

Sol. We know that, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$$\Rightarrow \frac{1}{6}\sin^3\theta = \frac{1}{8}\sin\theta - \frac{1}{24}\sin 3\theta$$

$$= \frac{1}{8}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) - \frac{1}{24}\left(3\theta - \frac{(3\theta)^3}{3!} + \frac{(3\theta)^5}{5!} - \frac{(3\theta)^7}{7!} + \ldots\right)$$

$$= \theta\left(\frac{1}{8} - \frac{3}{24}\right) + \frac{\theta^3}{3!}\left(\frac{3^3}{24} - \frac{1}{8}\right) - \frac{\theta^5}{5!}\left(\frac{3^5}{24} - \frac{1}{8}\right) + \frac{\theta^7}{7!}\left(\frac{3^7}{24} - \frac{1}{8}\right) - \ldots$$

$$= \frac{\theta^3}{3!} - \frac{\theta^5}{5!}\frac{3(3^2-1)(3^2+1)}{24} + \frac{\theta^7}{7!}\frac{3(3^2-1)}{24}(3^4+3^2+3) - \ldots$$

$$= \frac{\theta^3}{3!} - (1+3^2)\frac{\theta^5}{5!} + (1+3^2+3^4)\frac{\theta^7}{7!} - \ldots$$

Hence proved.

Q 19. **Prove that $32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$.**

(2006)

Sol. Let $x = \cos\theta + i\sin\theta$. Then, $\frac{1}{x} = \cos\theta - i\sin\theta$

So that, $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

$$\therefore \quad (2i\sin\theta)^4(2\cos\theta)^2 = \left\{x - \frac{1}{x}\right\}^4 \cdot \left\{x + \frac{1}{x}\right\}^2 \quad \ldots(i)$$

Binomial coefficients in the expansion of $\left[x - \left(\frac{1}{x}\right)\right]^4$ are 1, −4, 6, −4, 1.

So, the coefficient scheme is

1	−4	6	−4	1		
1	−3	2	2	−3	1	
1	−2	−1	4	−1	−2	1

From Eq. (i), we have

$$(2i\sin\theta)^4(2\cos\theta)^2 = x^6 - 2x^4 - x^2 + 4 - \left(\frac{1}{x^2}\right) - 2\left(\frac{1}{x^4}\right) + \left(\frac{1}{x^6}\right)$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - 1\left(x^2 + \frac{1}{x^2}\right) + 4$$

$$\Rightarrow \quad 2^6 i^4 \sin^4\theta\cos^2\theta = 2\cos 6\theta - 2\cdot 2\cos 4\theta - 2\cos 2\theta + 4$$

$$\Rightarrow \quad 2^5\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

[dividing both sides by 2]

$$\Rightarrow \quad 32\sin^4\theta\cos^2\theta = \cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2$$

Hence proved.

Q 20. Expand $\cos^7\theta$ in a series of cosines of multiple of θ. **(2005)**

Sol. Let $x = \cos\theta + i\sin\theta$, so that $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore\ x^n = \cos n\theta + i\sin n\theta$ and $\frac{1}{x^n} = \cos n\theta - i\sin n\theta$

Therefore, $x + \frac{1}{x} = 2\cos\theta,\ x^n + \frac{1}{x^n} = 2\cos n\theta$

We have, $(2\cos\theta)^7 = \left(x + \frac{1}{x}\right)^7 = x^7 + {}^7C_1 x^6 \cdot \frac{1}{x} + {}^7C_2 x^5 \cdot \frac{1}{x^2} + {}^7C_3 x^4 \cdot \frac{1}{x^3}$

$$+ {}^7C_4 x^3 \cdot \frac{1}{x^4} + {}^7C_5 x^2 \cdot \frac{1}{x^5} + {}^7C_6 x \cdot \frac{1}{x_6} + {}^7C_7 \cdot \frac{1}{x^7}$$

$$= \left(x^7 + \frac{1}{x^7}\right) + 7\left(x^5 + \frac{1}{x^5}\right) + 21\left(x^3 + \frac{1}{x^3}\right) + 35\left(x + \frac{1}{x}\right)$$

$$= 2\cos 7\theta + 7(2\cos 5\theta) + 21(2\cos 3\theta) + 35(2\cos\theta)$$

$$\Rightarrow \quad 2^7\cos^7\theta = 2(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$$

$$\Rightarrow \quad \cos^7\theta = \frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$$

Q 21. Prove that

$$\sin^2\theta\cos\theta = \theta^2 - \frac{5}{6}\theta^4 + \ldots + (-1)^{n+1}\frac{(3^{2n}-1)}{4}\frac{\theta^{2n}}{2n!} + \ldots . \quad \textbf{(2004)}$$

Or **Expand $\sin^2\theta\cos\theta$ in powers of θ and give the general term.** **(2010)**

Sol. We have, $\sin^2\theta\cos\theta = \sin\theta\sin\theta\cos\theta = \frac{1}{2}\sin\theta(2\sin\theta\cos\theta)$

$$= \frac{1}{2}\sin\theta\sin 2\theta = \frac{1}{4}(2\sin\theta\sin 2\theta) = \frac{1}{2}(\cos\theta - \cos 3\theta)$$

$$= \frac{1}{4}\left[\left\{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots + (-1)^n\frac{\theta^{2n}}{2n!} + \ldots\right\}\right.$$

$$\left. - \left\{1 - \frac{(3\theta)^2}{2!} + \frac{(3\theta)^4}{4!} + \ldots + (-1)^n\frac{(3\theta)^{2n}}{2n!} + \ldots\right\}\right]$$

$$= \frac{1}{4}\left[-\frac{\theta^2}{2!}(1-3^2) + \frac{\theta^4}{4!}(1-3^4) + \ldots + (-1)^n\frac{\theta^{2n}}{2n!}(1-3^{2n}) + \ldots\right]$$

$$= \frac{1}{4}\left[4\theta^2 - \frac{10}{3}\theta^4 + \ldots + (-1)^n(-1)(3^{2n}-1)\frac{\theta^{2n}}{2n!} + \ldots\right]$$

$$\therefore\ \sin^2\theta\cos\theta = \theta^2 - \frac{5}{6}\theta^4 + \ldots + (-1)^{n+1}\frac{(3^{2n}-1)}{4}\frac{\theta^{2n}}{2n!} + \ldots$$ **Hence proved.**

Q 22. Expand $\sin^7\theta\cos^2\theta$ in a series of sine of multiples of θ. (2012)

Sol. Let $x = \cos\theta + i\sin\theta$.

Then, $\frac{1}{x} = \cos\theta - i\sin\theta$, so that $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

$$\therefore \quad (2i\sin\theta)^7(2\cos\theta)^2 = \left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2$$

To obtain the coefficients of various powers of x in the product $\left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2$, we multiply $\left(x - \frac{1}{x}\right)^7$ in succession two times by $\left(x + \frac{1}{x}\right)$.

For this purpose, we make the following scheme

1	−7	21	−35	35	−21	7	−1		
1	−6	14	−14	0	14	−14	6	−1	
1	−5	8	0	−14	14	0	−8	5	−1

The last row in the above scheme gives the coefficients of the various powers of x in the product $\left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2$.

The highest power of x in the product $\left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2$ is 9 and then the power go on decreasing by 2.

$$\therefore \ (2i\sin\theta)^7(2\cos\theta)^2 = \left(x - \frac{1}{x}\right)^7\left(x + \frac{1}{x}\right)^2$$

$$= x^9 - 5x^7 + 8x^5 + 0\cdot x^3 - 14x + \frac{14}{x} + \frac{0}{x^3} - \frac{8}{x^5} + \frac{5}{x^7} - \frac{1}{x^9}$$

$$= \left(x^9 - \frac{1}{x^9}\right) - 5\left(x^7 - \frac{1}{x^7}\right) + 8\left(x^5 - \frac{1}{x^5}\right) - 14\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 9\theta - 5(2i\sin 7\theta) + 8(2i\sin 5\theta) - 14(2i\sin\theta)$$

$$\left[\because x^n - \frac{1}{x^n} = 2i\sin n\theta\right]$$

On dividing both sides by $2i$, we have

$$2^8 i^6 \sin^7\theta\cos^2\theta = \sin 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta$$

But $i^6 = (i^2)^3 = (-1)^3 = -1$, therefore

$$\sin^7\theta\cos^2\theta = -\frac{1}{2^8}(\sin 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta)$$

$$= -\frac{1}{256}(\sin 9\theta - 5\sin 7\theta + 8\sin 5\theta - 14\sin\theta)$$

Q 1. Use De-Moivre's theorem to solve the equation

$$x^7 + x^4 + x^3 + 1 = 0.$$ **(2009, 07)**

Sol. We have, $x^7 + x^4 + x^3 + 1 = 0 \Rightarrow x^4(x^3 + 1) + 1(x^3 + 1) = 0$

$\Rightarrow (x^3+1)(x^4+1) = 0 \Rightarrow x^3 + 1 = 0$ or $x^4 + 1 = 0 \Rightarrow x^3 = -1$ or $x^4 = -1$

On solving these equations separately, we have

$$x^3 = -1 = \cos(2r+1)\pi + i\sin(2r+1)\pi$$

$$\Rightarrow \quad x = \cos(2r+1)\frac{\pi}{3} + i\sin(2r+1)\frac{\pi}{3}, \ (r = 0, 1, 2)$$

The roots are given by $x_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$$x_1 = \cos\frac{3\pi}{3} + i\sin\frac{3\pi}{3} = \cos\pi + i\sin\pi = -1$$

and $$x_2 = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} \quad \left[\because \frac{5\pi}{3} = 2\pi - \frac{\pi}{3}\right]$$

$$= \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

The remaining four roots are given by

$$x^4 = -1 = \cos(2r+1)\pi + i\sin(2r+1)\pi$$

$$\Rightarrow \quad x = \cos(2r+1)\frac{\pi}{4} + i\sin(2r+1)\frac{\pi}{4}, \ r = 0, 1, 2, 3$$

$$\therefore \quad x_0' = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$x_1' = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$x_2' = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

and $$x_3' = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

Hence, the required roots are given by $x_0, x_1, x_2, x_0', x_1', x_2'$ and x_3'.

Q 2. Expand $\cos^6\theta \sin^3\theta$ in a series of sines of multiple of θ. **(2013)**

Sol. Let $x = \cos\theta + i\sin\theta$.

Then, $\frac{1}{x} = \cos\theta - i\sin\theta$, so that $x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

$$\therefore \quad (2\cos\theta)^6 (2i\sin\theta)^3 = \left(x + \frac{1}{x}\right)^6 \left(x - \frac{1}{x}\right)^3$$

To obtain the coefficients of various powers of x in the product $\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^3$, we multiply $\left(x+\frac{1}{x}\right)^6$ in succession three times by $\left(x-\frac{1}{x}\right)$.

For this purpose, we make the following scheme

1	6	15	20	15	6	1			
1	5	9	5	−5	−9	−5	−1		
1	4	4	−4	−10	−4	4	4	1	
1	3	0	−8	−6	6	8	0	−3	−1

The last row in the above scheme gives the coefficients of the various powers of x in the product $\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^3$.

The highest power of x in the product $\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^3$ is 9 and then the power go on decreasing by 2.

$$\therefore\ (2\cos\theta)^6\,(2i\sin\theta)^3=\left(x+\frac{1}{x}\right)^6\left(x-\frac{1}{x}\right)^3$$

$$=x^9+3x^7-8x^3-6x+\frac{6}{x}+\frac{8}{x^3}-\frac{3}{x^7}-\frac{1}{x^9}$$

$$=\left(x^9-\frac{1}{x^9}\right)+3\left(x^7-\frac{1}{x^7}\right)-8\left(x^3-\frac{1}{x^3}\right)-6\left(x-\frac{1}{x}\right)$$

$$\Rightarrow\ -i2^9\cos^6\theta\sin^3\theta=2i\sin 9\theta+3\,(2i\sin 7\theta)-8\,(2i\sin 3\theta)-6\,(2i\sin\theta).$$

$$\left[\because x^n-\frac{1}{x^n}=2i\sin n\theta\right]$$

On dividing both sides by $2i$, we get

$$-2^8\cos^6\theta\sin^3\theta=\sin 9\theta+3\sin 7\theta-8\sin 3\theta-6\sin\theta$$

$$\Rightarrow\quad \cos^6\theta\sin^3\theta=-\frac{1}{2^8}\,[\sin 9\theta+3\sin 7\theta-8\sin 3\theta-6\sin\theta]$$

$$=\frac{-1}{256}\,[\sin 9\theta+3\sin 7\theta-8\sin 3\theta-6\sin\theta]$$