

§ 8.1. INTRODUCTION

A general equation of second degree in x, y and z namely

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \quad \dots(1)$$

always represents a **conicoid** or a **quadratic surface**.

The equation (1) by a proper transformation of axes can be reduced to the following standard form

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(2)$$

The standard form (2) is called the **central conicoid**.

Properties of Central Conicoid :

(i) *The origin is the centre of the central conicoid*

Proof : Let $P(x_1, y_1, z_1)$ be any point on the central conicoid (2), so that we have

$$ax_1^2 + by_1^2 + cz_1^2 = 1$$

which can also be written as

$$a(-x_1)^2 + b(-y_1)^2 + c(-z_1)^2 = 1. \quad \dots(3)$$

The relation (3) shows that the point $Q(-x_1, -y_1, -z_1)$ will also lie on (2) if $P(x_1, y_1, z_1)$ lies on (2). The middle point of the chord PQ is $(0, 0, 0)$. This shows that all chords of the surface (2) which pass through the origin are bisected at the origin. Therefore the surface (2) has a centre and it is at the origin. Thus origin is the centre of the central conicoid (2) and due to this property the surface (2) is called a central conicoid.

(ii) *The co-ordinate planes bisect all chords perpendicular to them.*

Proof : Let $P(x_1, y_1, z_1)$ be a point on the central conicoid (2), then clearly the point $Q(-x_1, y_1, z_1)$ also lies on it. The middle point of the chord PQ is $(0, y_1, z_1)$ which clearly lies on the YOZ plane i.e. the plane $x = 0$. The d.r.'s of the chord PQ are $2x_1, 0, 0$ i.e. $1, 0, 0$ showing that the chord PQ is perpendicular to the plane YOZ . Hence we conclude that the co-ordinate plane YOZ bisects all chords perpendicular to it. Similarly we can prove that the co-ordinate planes ZOX and XOY also bisect all chords perpendicular to them respectively.

Hence the central conicoid (2) is symmetrical with respect to all the three co-ordinate planes and these planes are called the **principal planes** of the central conicoid.

The three principal planes (*i.e.* the co-ordinate planes in this case) taken in pairs intersect in three lines (*i.e.* the co-ordinate axes in this case) which are called the principal axes of the central conicoid.

(iii) The standard form (2) of the central conicoid can take three forms each of which represents a surface with a definite name assigned to it. Which we shall discuss in detail in Articles 8.2, 8.3 & 8.4 given below.

§ 8.2. THE ELLIPSOID

The standard equation of the ellipsoid is given by

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad \dots(4)$$

Properties :

(i) The origin is the centre of the ellipsoid given by (4).

(ii) The co-ordinate planes bisect all chords perpendicular to them.

(iii) The ellipsoid is a closed surface. The equation (4) may be rewritten as

$$y^2/b^2 + z^2/c^2 = 1 - x^2/a^2. \quad \dots(5)$$

If x is numerically greater than a , then from the equation (5) we find that $y^2/b^2 + z^2/c^2$ is a negative quantity *i.e.* the sum of two perfect squares is a negative quantity which shows that at least one of y and z should be imaginary. Hence it follows that x can not be numerically greater than a and therefore, the surface exists between the two parallel planes $x = a$ and $x = -a$. In a similar manner it follows that the surface (4) lies between the parallel planes $y = b, y = -b$ and $z = c, z = -c$.

Thus the ellipsoid (4) is a closed surface.

(iv) The intercepts on the co-ordinate axes. The equation of the x -axis are $y = 0, z = 0$. It meets the surface (4) in the points $A(a, 0, 0)$ and $A'(-a, 0, 0)$. Therefore, the surface (4) intercepts a length $2a$ on the x -axis. In a similar manner the surface (4) intercepts lengths $2b$ and $2c$ on the y and z axes respectively. These intercepts $2a, 2b$ and $2c$ are called the lengths of the axes of the ellipsoid.

(v) The sections of the ellipsoid by the planes parallel to the co-ordinate planes.

The section of the ellipsoid by the plane $z = k$, parallel to xy -plane is given by the equations :

$$z = k, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{k^2}{c^2}$$

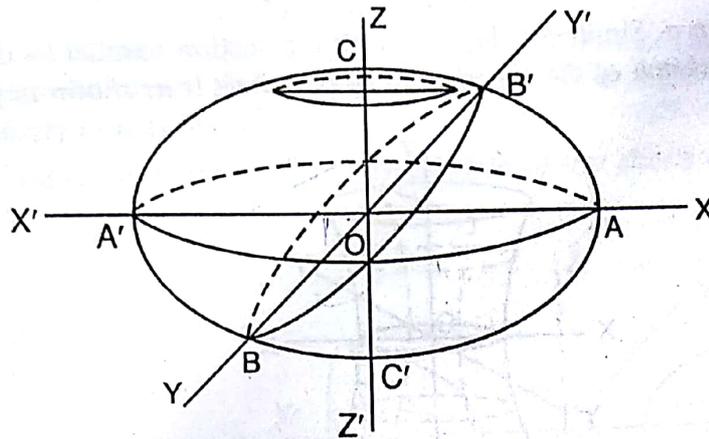
This is an ellipse with its centre on the z -axis and the axes parallel to the x and y axes. As k varies from $-c$ to 0 , the ellipse increases in size from the point-ellipse

at C' to ellipse $z = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and as k varies from 0 to c it again diminishes to

a point-ellipse at C . When $k > c$ or $k < -c$, the ellipse is imaginary. The surface is thus generated by a variable ellipse and hence it is called an ellipsoid.

Similarly the sections of the ellipsoid by the planes parallel to yz and zx planes are ellipses.

(vi) A rough sketch of the ellipsoid is as shown in fig.



8.3. THE HYPERBOLOID OF ONE SHEET

The standard equation of the hyperboloid of one sheet is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \dots(6)$$

Properties :

(i) All the chords passing through the origin are bisected at the origin. Therefore, the surface (6) has the origin as its centre.

(ii) The co-ordinate planes bisect all chords perpendicular to them. Hence we can say that the hyperboloid of one sheet (6) is symmetrical with respect to all the three co-ordinate planes and these planes are called the principal planes of the surface (6).

(iii) The intercepts on the co-ordinate axes. The equations of the x-axis are $y = 0, z = 0$. It meets the surface (6) in the points A $(a, 0, 0)$ and A' $(-a, 0, 0)$. Therefore, the surface (6) intercepts a length $2a$ on the x-axis. In a similar manner the surface (6) intercepts a length $2b$ on the y-axis. The z-axis i.e. $x = 0, y = 0$ meets the surface (6) in imaginary points $[z = \pm \sqrt{-c}]$ i.e. the z-axis does not meet this surface (6) in real points.

(iv) The sections of the hyperboloid of one sheet by the planes parallel to the co-ordinate planes.

The section of the hyperboloid of one sheet by a plane $z = k$, parallel to the xy-plane, is given by the equations.

$$z = k, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}$$

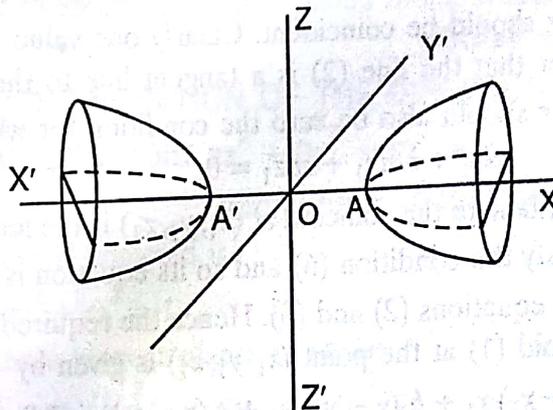
This is an ellipse whose size goes on increasing as k increases numerically. The smallest ellipse corresponding to $k = 0$ is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ and as k tends to $+\infty$ or $-\infty$ the elliptic section expands to infinite dimensions.

The section of the hyperboloid of one sheet by the plane $x = k$, parallel to the yz-plane is the hyperbola $x = k, \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{k^2}{a^2}$.

$$x = k, \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1$$

which is real only when k is numerically greater than a . There is no part of the surface between the planes $x = -a$ and $x = a$, but beyond these planes the surface extends to infinity in both directions.

(v) A rough sketch of the hyperboloid of two sheets is as given in fig.



Concluding Results : From the theory discussed above we can say that the standard equation $ax^2 + by^2 + cz^2 = 1$ of the central conicoid represents.

- (i) An ellipsoid if a, b, c are all positive.
- (ii) A hyperboloid of one sheet if any two of a, b, c are positive and the remaining third is negative, and
- (iii) A hyperboloid of two sheets if any two of a, b and c are negative and the remaining third is positive.

§ 8.5. EQUATION OF THE TANGENT PLANE TO THE CENTRAL CONICOID

$$ax^2 + by^2 + cz^2 = 1 \text{ AT THE POINT } (x_1, y_1, z_1).$$

The equation of the central conicoid is

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

Let (x_1, y_1, z_1) be any given point on the surface (1).

The equations of any line through (x_1, y_1, z_1) and having d.c.'s l, m, n are given

by

$$(x - x_1)/l = (y - y_1)/m = (z - z_1)/n = r \text{ (say)} \quad \dots(2)$$

The co-ordinates of any point on the line (2) are $(lr + x_1, mr + y_1, nr + z_1)$. Let the line (2) meet the conicoid (1) at this point, then this point will satisfy the equation (1) and so we have

$$a(lr + x_1)^2 + b(mr + y_1)^2 + c(nr + z_1)^2 = 1$$

or

$$r^2 (al^2 + bm^2 + cn^2) + 2r(ax_1l + by_1m + cz_1n) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \dots(3)$$

Since the point (x_1, y_1, z_1) lies on the conicoid (1), we have

$$ax_1^2 + by_1^2 + cz_1^2 = 1. \quad \dots(4)$$

Using (4), the equation (3) becomes

$$r^2 (a^2 + b^2 + c^2) + 2r (ax_1 + by_1 + cz_1) = 0 \quad \dots(5)$$

The equation (5) is a quadratic equation in r and hence gives two values of r i.e. the line (2) meets the surface (1) in two points.

If the line (2) is a tangent line to the conicoid (1) at the point (x_1, y_1, z_1) then the two values of r should be coincident. Clearly one value of r given by (5) is zero and hence in order that the line (2) is a tangent line to the conicoid at (x_1, y_1, z_1) , the other value of r should also be zero the condition for which is

$$ax_1 + by_1 + cz_1 = 0. \quad \dots(6)$$

The tangent plane to the conicoid at (x_1, y_1, z_1) is the locus of such lines through (x_1, y_1, z_1) that satisfy the condition (6) and so its equation is obtained by eliminating l, m, n between the equations (2) and (6). Hence the required equation of the tangent plane to the conicoid (1) at the point (x_1, y_1, z_1) is given by

$$a(x - x_1)x_1 + b(y - y_1)y_1 + c(z - z_1)z_1 = 0$$

$$\begin{aligned} \text{or} \quad & axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2 \\ \text{or} \quad & axx_1 + byy_1 + czz_1 = 1. \end{aligned} \quad \text{using (4)}$$

Working rule for problems : To write the tangent plane to the conicoid at the point (x_1, y_1, z_1) replace x^2 by xx_1 , y^2 by yy_1 and z^2 by zz_1 .

§ 8.6. DEDUCTION OF THE CONDITION THAT THE PLANE

$lx + my + nz = p$ MAY TOUCH THE CENTRAL CONICOID

$$ax^2 + by^2 + cz^2 = 1.$$

The equation of the given plane is

$$lx + my + nz = p \quad \dots(1)$$

and the equation of the given conicoid is

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(2)$$

Let the plane (1) touch the conicoid (2) at the point (x_1, y_1, z_1) . The equation of the tangent plane to the conicoid (2) at the point (x_1, y_1, z_1) is

$$axx_1 + byy_1 + czz_1 = 1. \quad \dots(3)$$

If the plane (1) touches the conicoid (2) at the point (x_1, y_1, z_1) , then the equations (1) and (3) represent the same plane and hence comparing their coefficients, we have

$$\frac{l}{ax_1} = \frac{m}{by_1} = \frac{n}{cz_1} = \frac{p}{1} \quad \text{or} \quad x_1 = \frac{l}{ap}, y_1 = \frac{m}{bp}, z_1 = \frac{n}{cp}. \quad \dots(4)$$

Again since (x_1, y_1, z_1) lies on (2), we have

$$ax_1^2 + by_1^2 + cz_1^2 = 1$$

$$\text{or} \quad a(l/ap)^2 + b(m/bp)^2 + c(n/cp)^2 = 1, \quad \text{using (4)}$$

or
$$l^2/a + m^2/b + n^2/c = p^2. \dots(5)$$

This is the required condition that the plane (1) touches the conicoid (2).

The co-ordinates of the point of contact of the plane (1) with the conicoid (2) are given by (4) i.e. are $(l/ap, m/bp, n/cp)$ where p is given by (5).

§ 8.7. THE DIRECTOR SPHERE :

Definition. The director sphere of a central conicoid is the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid.

(Gorakhpur 2011, 15)

§ 8.7.1. DEDUCTION OF THE EQUATION OF THE DIRECTOR SPHERE

OF THE CENTRAL CONICOID $ax^2 + by^2 + cz^2 = 1.$

(Gorakhpur 2006; Purvanchal 2006)

The equation of the central conicoid is

$$ax^2 + by^2 + cz^2 = 1. \dots(1)$$

Let the equations of three tangent planes to (1) be

$$l_1x + m_1y + n_1z = \sqrt{\{l_1^2/a + m_1^2/b + n_1^2/c\}} \dots(2)$$

$$l_2x + m_2y + n_2z = \sqrt{\{l_2^2/a + m_2^2/b + n_2^2/c\}} \dots(3)$$

and
$$l_3x + m_3y + n_3z = \sqrt{\{l_3^2/a + m_3^2/b + n_3^2/c\}} \dots(4)$$

where $l_1, m_1, n_1; l_2, m_2, n_2$ and l_3, m_3, n_3 are the actual direction cosines of the normals to the above three tangent planes respectively. If these tangent planes are mutually perpendicular, then $l_1, l_2, l_3; m_1, m_2, m_3$ and n_1, n_2, n_3 are also dc's of any three mutually perpendicular lines [Refer art. 4.4]. Therefore, we have

$$l_1^2 + l_2^2 + l_3^2 = 1, m_1^2 + m_2^2 + m_3^2 = 1, n_1^2 + n_2^2 + n_3^2 = 1 \dots(5)$$

$$l_1m_1 + l_2m_2 + l_3m_3 = 0, m_1n_1 + m_2n_2 + m_3n_3 = 0, \dots(6)$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0.$$

The director sphere of the surface (1), being the locus of the point of intersection of the three tangent planes (2), (3) and (4) is obtained by eliminating $l_1, m_1, n_1; l_2, m_2, n_2$ and l_3, m_3, n_3 between the equations (2), (3) and (4) with the help of the relations (5) and (6). Squaring and adding (2), (3) and (4), we get

$$(l_1x + m_1y + n_1z)^2 + (l_2x + m_2y + n_2z)^2 + (l_3x + m_3y + n_3z)^2 = (1/a) \Sigma l_1^2 + (1/b) \Sigma m_1^2 + (1/c) \Sigma n_1^2$$

or
$$x^2 \Sigma l_1^2 + y^2 \Sigma m_1^2 + z^2 \Sigma n_1^2 + 2yz \Sigma m_1n_1 + 2zx \Sigma n_1l_1 + 2xy \Sigma l_1m_1 = (1/a) + (1/b) + (1/c),$$
 using (5)

or
$$x^2 + y^2 + z^2 = (1/a) + (1/b) + (1/c),$$
 using (5) and (6).

This is the required equation of the director sphere of the central conicoid (1).

SOLVED EXAMPLES (SET A)

Example 1. Show that the plane $x + 2y + 3z = 2$ touches the conicoid

$$x^2 - 2y^2 + 3z^2 = 2.$$

(Gorakhpur 2004, 09, 16)

Sol. The equation of the given conicoid is

$$x^2 - 2y^2 + 3z^2 = 2 \text{ or } (1/2)x^2 - y^2 + (3/2)z^2 = 1. \quad \dots(1)$$

The equation of the given plane is

$$x + 2y + 3z = 2. \quad \dots(2)$$

If the plane (2) touches the conicoid (1), then applying the condition

$$l^2/a + m^2/b + n^2/c = p^2 \text{ [See Art. 8.6], we have}$$

$$\frac{1}{1/2} + \frac{4}{-1} + \frac{9}{3/2} = 4 \text{ or } 2 - 4 + 6 = 4 \text{ or } 4 = 4.$$

Which is true. Hence the given plane (2) touches the given conicoid.

Example 2. Find the equations to the tangent planes to the hyperboloid

$$2x^2 - 6y^2 + 3z^2 = 5 \text{ which pass through the line}$$

$$3x - 3y + 6z - 5 = 0 = x + 9y - 3z.$$

Sol. The equation of the given hyperboloid is

$$2x^2 - 6y^2 + 3z^2 = 5 \text{ or } (2/5)x^2 - (6/5)y^2 + (3/5)z^2 = 1. \quad \dots(1)$$

The equations of the given line are

$$3x - 3y + 6z - 5 = 0, x + 9y - 3z = 0. \quad \dots(2)$$

The equation of any plane through the line (2) is

$$3x - 3y + 6z - 5 + \lambda(x + 9y - 3z) = 0$$

or $(3 + \lambda)x + (-3 + 9\lambda)y + (6 - 3\lambda)z = 5. \quad \dots(3)$

If the plane (3) touches the hyperboloid (1), then applying the condition

$$l^2/a + m^2/b + n^2/c = p^2 \text{ [See Article 8.6], we have}$$

$$(5/2)(3 + \lambda)^2 + (-5/6)(-3 + 9\lambda)^2 + (5/3)(6 - 3\lambda)^2 = (5)^2$$

or $15(9 + 6\lambda + \lambda^2) - 5(9 - 54\lambda + 81\lambda^2) + 10(36 - 36\lambda + 9\lambda^2) = 150$

or $3(9 + 6\lambda + \lambda^2) - (9 - 54\lambda + 81\lambda^2) + 2(36 - 36\lambda + 9\lambda^2) = 30$

or $-60\lambda^2 + 60 = 0 \text{ or } \lambda^2 = 1 \text{ or } \lambda = \pm 1.$

When $\lambda = 1$, from (3) the equation of the tangent plane is

$$4x + 6y + 3z = 5.$$

When $\lambda = -1$, from (3) the equation of the tangent plane is

$$2x - 12y + 9z = 5.$$

Example 3. A tangent plane to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ meets the co-ordinate axes in points P, Q and R. Prove that the centroid of the triangle PQR lies on the surface

$$a^2/x^2 + b^2/y^2 + c^2/z^2 = 9.$$

Sol. The equation of the given ellipsoid is

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad \dots(1)$$

The equation of any tangent plane to the ellipsoid (1) is

$$lx + my + nz = \sqrt{a^2l^2 + b^2m^2 + c^2n^2}. \quad \dots(2)$$

The plane (2) meets the co-ordinate axes in the points given by

$$P \{ \sqrt{a^2l^2 + b^2m^2 + c^2n^2}/l, 0, 0 \}; Q \{ 0, \sqrt{a^2l^2 + b^2m^2 + c^2n^2}/m, 0 \}$$

and $R \{ 0, 0, \sqrt{a^2l^2 + b^2m^2 + c^2n^2}/n \}.$

Let (α, β, γ) be the co-ordinates of the centroid of the triangle PQR , then

$$\alpha = \frac{\sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}/l + 0 + 0}{3} = \frac{1}{3l} \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)},$$

$$\beta = \frac{1}{3m} \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}, \gamma = \frac{1}{3n} \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}.$$

Thus $3l\alpha = \sqrt{(a^2l^2 + b^2m^2 + c^2n^2)}$ or $9l^2\alpha^2 = a^2l^2 + b^2m^2 + c^2n^2$.

$$\therefore \frac{a^2}{\alpha^2} = \frac{9l^2a^2}{9l^2\alpha^2} = \frac{9a^2l^2}{a^2l^2 + b^2m^2 + c^2n^2}$$

$$\text{Similarly } \frac{b^2}{\beta^2} = \frac{9b^2m^2}{a^2l^2 + b^2m^2 + c^2n^2}, \frac{c^2}{\gamma^2} = \frac{9c^2n^2}{a^2l^2 + b^2m^2 + c^2n^2} \dots(3)$$

Adding the relations in (3), we get

$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = \frac{9(a^2l^2 + b^2m^2 + c^2n^2)}{a^2l^2 + b^2m^2 + c^2n^2} = 9.$$

\therefore The locus of the centroid (α, β, γ) is, $a^2/x^2 + b^2/y^2 + c^2/z^2 = 9$.

§ 8.8. THE POLAR PLANE

(Gorakhpur 2013, 15)

Definition. Let $ax^2 + by^2 + cz^2 = 1$... (1)

be the equation of a conicoid and $A(\alpha, \beta, \gamma)$ be any point. Draw lines APQ to meet the conicoid (1) in the points P and Q . The locus of the point R such that AR is the harmonic mean of AP and AQ (i.e., AP, AR and AQ are in harmonic progression) is called the polar plane of the point A with respect to the given conicoid and A is called its pole.

§ 8.8.1. Deduction of the equation of the polar plane

(Gorakhpur 2013, 15)

The equations of any line through the point $A(\alpha, \beta, \gamma)$ are

$$(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n = r \text{ (say)}, \dots(2)$$

where l, m, n are the actual d.c.'s of the line.

The co-ordinates of any point on (2) distant r from $A(\alpha, \beta, \gamma)$ are $(lr + \alpha, mr + \beta, nr + \gamma)$. Therefore, the distances of the points P and Q where the line (2) meets the given conicoid (1) are given by the quadratic in r^2

$$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 = 1$$

$$\text{or } r^2(a^2l^2 + b^2m^2 + c^2n^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0. \dots(3)$$

Let the two values of r be r_1 and r_2 and let $r_1 = AP$ and $r_2 = AQ$. Now AP, AR, AQ are in harmonic progression.

$$\therefore \frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1 + r_2}{r_1r_2}$$

$$\text{or } \frac{2}{AR} = \frac{-2(al\alpha + bm\beta + cn\gamma)}{a\alpha^2 + b\beta^2 + c\gamma^2 - 1}$$

or $ax^2 + by^2 + cz^2 - 1 = - \{a\alpha (l \cdot AR) + b\beta (m \cdot AR) + c\gamma (n \cdot AR)\}$... (4)
 Now let (x, y, z) be the co-ordinates of the point R on the line (1) whose distance from the point $A (\alpha, \beta, \gamma)$ is AR and so we have

$$(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n = AR.$$

$$\therefore l \cdot AR = x - \alpha, m \cdot AR = y - \beta, n \cdot AR = z - \gamma.$$

Now in order to find the locus of R , we are to eliminate l, m, n between (4) and (5) and so it is given by

$$ax^2 + by^2 + cz^2 - 1 = - \{a\alpha (x - \alpha) + b\beta (y - \beta) + c\gamma (z - \gamma)\}$$

or $axx + b\beta y + c\gamma z = 1.$

This is the required equation of the polar plane of the point $A (\alpha, \beta, \gamma)$ with respect to the conicoid (1). ... (6)

§ 8·8·2. Deduction of the pole of a given plane.

To find the pole of the plane $lx + my + nz = p$ w.r.t. the conicoid

$$ax^2 + by^2 + cz^2 = 1.$$

The equation of the given conicoid is

$$ax^2 + by^2 + cz^2 = 1. \quad \dots (1)$$

Let the pole of the plane

$$lx + my + nz = p \quad \dots (2)$$

w.r.t. the conicoid (1) be the point $P (\alpha, \beta, \gamma)$. But the polar plane of $P (\alpha, \beta, \gamma)$ w.r.t. to (1) is

$$a\alpha x + b\beta y + c\gamma z = 1. \quad \dots (3)$$

Therefore the equations (2) and (3) represent the same plane and so comparing their coefficients, we get

$$\frac{a\alpha}{l} = \frac{b\beta}{m} = \frac{c\gamma}{n} = \frac{1}{p} \text{ i.e. } \alpha = \frac{l}{ap}, \beta = \frac{m}{bp}, \gamma = \frac{n}{cp}.$$

\therefore The required pole is $l/(ap), m/(bp), n/(cp)$.

§ 8·9. PROPERTIES OF THE POLAR PLANES AND THE POLAR LINES

(1) Let $P (\alpha_1, \beta_1, \gamma_1)$ and $Q (\alpha_2, \beta_2, \gamma_2)$ be any two points. It clearly follows that if the polar plane of the point P with respect to a given conicoid passes through the point Q then the polar plane of Q w.r.t. the same conicoid will pass through P . The points P and Q satisfying this property are called the **conjugate points**.

(2) Let $u_1 \equiv l_1x + m_1y + n_1z - p_1 = 0$

and $u_2 \equiv l_2x + m_2y + n_2z - p_2 = 0$

be the equations of the two planes. It clearly follows that if the pole of the plane $u_1 = 0$ lies on the plane $u_2 = 0$ then the pole of the plane $u_2 = 0$ will lie on the plane $u_1 = 0$. The two such planes $u_1 = 0$ and $u_2 = 0$ are called the **conjugate planes**.

(3) **The polar lines.** The polar line or simply polar of a given line AB is another line PQ such that the polar planes of all points on AB pass through PQ .

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

Let the equations of the line AB be

$$(x - \alpha)/l = (y - \beta)/m = (z - \gamma)/n = r \text{ (say)}. \quad \dots(2)$$

Any point on (2) is $R(lr + \alpha, mr + \beta, nr + \gamma)$. The equation of the polar plane of the point R with respect to the conicoid (1) is

$$ax(lr + \alpha) + by(mr + \beta) + cz(nr + \gamma) = 1$$

or $(acx + b\beta y + c\gamma z - 1) + r(ax + bmy + cnz) = 0.$

This plane for all values of r clearly passes through the line of intersection of the fixed planes

$$acx + b\beta y + c\gamma z - 1 = 0, ax + bmy + cnz = 0.$$

Let this line be the line PQ .

Thus we observe that the polar plane of any point R on AB passes through every point of the line PQ and hence the polar plane of any point on the line PQ must pass through R . But R is an arbitrary point on AB and so the polar plane of every point on PQ will pass through every point of the line AB .

Thus we conclude that the lines AB and PQ are such that the polar plane of every point on AB passes through PQ and vice versa and the two such lines are called the *polar lines*.

(4) To find the condition that the two given lines be the polar lines.

Let the equation of the conicoid be

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

Let the two given lines be denoted by AB and PQ and let their equations be given by

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

and $\frac{x - \alpha'}{l'} = \frac{y - \beta'}{m'} = \frac{z - \gamma'}{n'}$ respectively.

Proceeding as in (3) above the polar line of AB with respect to the conicoid (1) is the line of intersection of the two planes given by

$$acx + b\beta y + c\gamma z - 1 = 0 \quad \dots(2)$$

and $ax + bmy + cnz = 0. \quad \dots(3)$

Now if the polar line of AB be the given line PQ then PQ will lie on the planes (2) and (3) both. The conditions for which are

$$\left. \begin{aligned} a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' - 1 &= 0 \\ a\alpha l' + b\beta m' + c\gamma n' &= 0 \end{aligned} \right\} \quad \dots(4)$$

and $\left. \begin{aligned} a\alpha\alpha' + b\beta\beta' + c\gamma\gamma' &= 0 \\ a\alpha l' + b\beta m' + c\gamma n' &= 0 \end{aligned} \right\} \quad \dots(5)$

Hence the required conditions are given by (4) and (5).

(5) **Conjugate lines.** Two lines AB and PQ are called the *conjugate lines* if when the line PQ intersects the polar line of the line AB then the line AB also intersects the polar line of the line PQ .

§ 8·11. NORMAL TO A CONICOID.

8·11·1. Deduction of the equations of the normal to a central conicoid at the point (α, β, γ) .

(Gorakhpur 2009)

Let the equation of the central conicoid be

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

The equation of the tangent plane at (α, β, γ) of the conicoid (1) is

$$a\alpha x + b\beta y + c\gamma z = 1. \quad \dots(2)$$

The normal to the conicoid (1) at (α, β, γ) is the straight line perpendicular to the tangent plane (2) and passing through (α, β, γ) and hence the required equations of the normal are given by

$$(x - \alpha)/(a\alpha) = (y - \beta)/(b\beta) = (z - \gamma)/(c\gamma) \quad \dots(3)$$

where $a\alpha, b\beta, c\gamma$ are the direction ratios of the normal (3).

Now let p be the length of the perpendicular from the origin to the tangent plane (2) so that

$$p = \frac{1}{\sqrt{(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)}}$$

or $(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1. \quad \dots(4)$

In view of (4) the actual direction cosines of the normal (3) are $a\alpha p, b\beta p, c\gamma p$ and hence the equations (3) of the normal to the conicoid (1) at (α, β, γ) in terms of actual direction cosines are given by

$$\frac{x - \alpha}{a\alpha p} = \frac{y - \beta}{b\beta p} = \frac{z - \gamma}{c\gamma p}. \quad \dots(5)$$

§ 8.12. NUMBER OF NORMALS

8.12.1. Six normals can be drawn to an ellipsoid from a given point (x_1, y_1, z_1) . (Gorakhpur 2004, 10, 16)

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \dots(1)$$

The equations of the normal to the ellipsoid (1) at $P(\alpha, \beta, \gamma)$ are

$$\frac{x - \alpha}{\alpha/a^2} = \frac{y - \beta}{\beta/b^2} = \frac{z - \gamma}{\gamma/c^2} = \lambda \text{ (say)}. \quad \dots(2)$$

If the normal (2) passes through the given point (x_1, y_1, z_1) , then this point will satisfy (2) and hence we have

$$\frac{x_1 - \alpha}{\alpha/a^2} = \frac{y_1 - \beta}{\beta/b^2} = \frac{z_1 - \gamma}{\gamma/c^2} = \lambda.$$

$$\therefore x_1 = \alpha + \frac{\alpha\lambda}{a^2}, y_1 = \beta + \frac{\beta\lambda}{b^2}, z_1 = \gamma + \frac{\gamma\lambda}{c^2}$$

$$\text{or } \frac{\alpha}{a} = \frac{\alpha x_1}{a^2 + \lambda}, \frac{\beta}{b} = \frac{\beta y_1}{b^2 + \lambda}, \frac{\gamma}{c} = \frac{\gamma z_1}{c^2 + \lambda} \quad \dots(3)$$

But the point $P(\alpha, \beta, \gamma)$ lies on the ellipsoid (1).

$$\therefore \alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2 = 1$$

$$\text{or } \frac{a^2 x_1^2}{(a^2 + \lambda)^2} + \frac{b^2 y_1^2}{(b^2 + \lambda)^2} + \frac{c^2 z_1^2}{(c^2 + \lambda)^2} = 1 \quad \dots(4)$$

[Putting the values from (3)]

This equation is of sixth degree in λ and hence gives six values of λ . Putting these six values of λ one by one in (3), we get six values of α, β, γ i.e. we get six points on the ellipsoid (1) the normals at which pass through a given point (x_1, y_1, z_1) .

Hence from a given point six normals can be drawn to an ellipsoid.

§ 8.14 ENVELOPING CONE

Definition. The locus of tangent lines to a conicoid through any given point is called an enveloping cone.

8.14.1 Deduction of equation of enveloping cone of central conicoid

Let the equation of a central conicoid be $ax^2 + by^2 + cz^2 = 1$. Suppose

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \quad \dots(1)$$

be equation of a line passing through a given point (α, β, γ) .

§ 8.15 ENVELOPING CYLINDER

Definition : The locus of the tangents to the conicoid, which are parallel to a given line is, called enveloping cylinder. (Gorakhpur 2009)

8.15.1 Deduction of equation of enveloping cylinder of central conicoid :

Let the equation of a central conicoid be (Gorakhpur 2009)

$$ax^2 + by^2 + cz^2 = 1 \quad \dots(1)$$

and the tangent lines of this conicoid are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. Then the equation of a tangent line passing through any point (α, β, γ) can be written as

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

This line touch the central conicoid (1) if [Equation (3) of art 8.14.1]

$$(a\alpha l + b\beta m + c\gamma n)^2 = (al^2 + bm^2 + cn^2) (a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

Therefore locus of (α, β, γ) is

$$(alx + bmy + czz)^2 = (al^2 + bm^2 + cn^2) (ax^2 + by^2 + cz^2 - 1)$$

which is the required equation of enveloping cylinder.

§ 8·16. DIAMETRICAL PLANE

Definition. For a central conicoid, a diametral plane is the locus of the middle points of a system of parallel chords.

8·16·1. Deduction of equation of a diametral plane.

Let the equation of a central conicoid be

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

Let the equations of a system of parallel chords be

$$(x - a)/l = (y - \beta)/m = (z - \gamma)/n = r \quad (\text{say}), \quad \dots(2)$$

where l, m, n are the direction cosines of each chord and so are fixed, and r is the actual distance of any point $(\alpha + lr, \beta + mr, \gamma + nr)$ on (2) from (α, β, γ) . If (2) meets (1) then the distances of the points of intersection of (1) and (2) from (α, β, γ) are the roots of the quadratic in r given by

$$a(\alpha + lr)^2 + b(\beta + mr)^2 + c(\gamma + nr)^2 = 1$$

or

$$r^2(al^2 + bm^2 + cn^2) + 2r(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \quad \dots(3)$$

If (α, β, γ) be the middle point of the chord, then its distances from the two points of intersection should be equal in magnitude and opposite in sign and hence the sum of the two values of r should be zero, so that we have

$$a\alpha + b\beta + c\gamma = 0$$

\therefore The locus of the middle points (α, β, γ) of the system of parallel chords with d.c.'s l, m, n i.e. the equation of a diametral plane for the conicoid (1) is

$$alx + bmy + cnz = 0. \quad \dots(4)$$

Clearly the diametral plane (4) passes through the centre $(0, 0, 0)$ of the conicoid (1).

If l, m, n are the d.r.'s of parallel chords even then the equation of the corresponding diametral plane is given by (4).

8.16.2. Every plane through the centre is a diametral plane of the central conicoid corresponding to some direction.

Let the equation of the central conicoid be

$$ax^2 + by^2 + cz^2 = 1. \quad \dots(1)$$

Consider a plane $Ax + By + Cz = 0$... (2)

through the central $(0, 0, 0)$ of the conicoid (1).

We have to prove that the plane (2) is a diametral plane of (1) corresponding to some direction l, m, n (say).

Now the equation of the diametral plane of (1) corresponding to the direction l, m, n is

$$alx + bmy + cnz = 0. \quad \dots(3)$$

The planes (2) and (3) should be the same and hence comparing them, we get

$$\frac{al}{A} = \frac{bm}{B} = \frac{cn}{C} \quad \text{or} \quad \frac{l}{A/a} = \frac{m}{B/b} = \frac{n}{C/c}$$

Hence the plane (2) is a diametral plane of the conicoid (1) corresponding to the direction l, m, n given by (4)

§ 8.17. CONJUGATE DIAMETERS AND CONJUGATE DIAMETRAL PLANES

Let the equation of the ellipsoid be

(Gorakhpur 2009)

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 \quad \dots(1)$$

with its centre at $O(0, 0, 0)$.

Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid (1) so that d.r.'s of the line OP are x_1, y_1, z_1 . The equation of the diametral plane [See 8.14.1] of the line OP is

$$xx_1/a^2 + yy_1/b^2 + zz_1/c^2 = 0. \quad \dots(2)$$

Now take a point $Q(x_2, y_2, z_2)$ on the section of the ellipsoid (1) by the plane (2). This point Q will satisfy both the equations of the ellipsoid (1) and plane (2).

$$\therefore xx_2/a^2 + yy_2/b^2 + zz_2/c^2 = 0. \quad \dots(3)$$

Thus the relation (3) is the condition that the diametral plane of the line OP passes through another point Q on the ellipsoid (1). But the result (3) is symmetrical in x_1, y_1, z_1 and x_2, y_2, z_2 and hence it shows that the diametral plane of the line OQ will also pass through the point P .

Thus we conclude that if the diametral plane of a line OP passes through the point Q then the diametral plane of the line OQ will also pass through the point P .

Now let the line of intersection of the diametral planes of OP and OQ cut the surface of the given ellipsoid in the point $R(x_3, y_3, z_3)$, so that R lies on both the diametral planes of OP and OQ and consequently the points P and Q lie on the diametral plane of the line OR i.e. on the plane.

$$xx_3/a^2 + yy_3/b^2 + zz_3/c^2 = 0. \quad \dots(4)$$

Hence it shows that the diametral plane of OR is the plane POQ . In a similar way the diametral planes of OP and OQ are the planes QOR and ROP respectively.

8·17·1. Conjugate semi-diameters. The three semi-diameters OP , OQ and OR of an ellipsoid which are such that the plane containing any two is the diametral plane of the third, are called the conjugate semi-diameters.

8·17·2. Conjugate planes. The three planes POQ , QOR and ROP which are such that each is the diametral plane of the line of intersection of the other two, are called the conjugate planes or conjugate diametral planes.

4), (2) ...

8.19. PROPERTIES OF CONJUGATE SEMI-DIAMETERS OF AN ELLIPSOID

I. The sum of the squares of any three conjugate semi-diameters of an ellipsoid is constant. (Gorakhpur 2009)

Proof : We have $OP^2 + OQ^2 + OR^2$

$$= (x_1^2 + y_1^2 + z_1^2) + (x_2^2 + y_2^2 + z_2^2) + (x_3^2 + y_3^2 + z_3^2)$$

$$= (x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2) + (z_1^2 + z_2^2 + z_3^2)$$

$$= a^2 + b^2 + c^2$$

[using relations (4) of 8.18]

which is a constant. Hence proved.

Example 1. Let OP, OQ, OR be the conjugate semi-diameters of the ellipsoid $3x^2 + \lambda y^2 + z^2 = 1$. Find λ if $OP^2 + OQ^2 + OR^2 = 2$. (Gorakhpur 2010)

Sol. The equation of the given ellipsoid is

$$3x^2 + \lambda y^2 + z^2 = 1.$$

$$\therefore a^2 = \frac{1}{3}, b^2 = 1/\lambda, c^2 = 1.$$

We know that $OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$.

[See property I of 8.19]

$$\therefore 2 = \frac{1}{3} + 1/\lambda + 1 \text{ or } \lambda = 3/2.$$

Example 2. Find the equation of the plane PQR .

(Gorakhpur 2014)

Sol. Let the equation of the plane PQR be

$$lx + my + nz = p. \quad \dots(1)$$

Since the plane (1) passes through $P(x_1, y_1, z_1), Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$,

we have

$$lx_1 + my_1 + nz_1 = p, \quad \dots(2)$$

$$lx_2 + my_2 + nz_2 = p, \quad \dots(3)$$

and $lx_3 + my_3 + nz_3 = p, \quad \dots(4)$

Multiplying the equations (2), (3) and (4) by x_1, x_2 and x_3 respectively and adding, we have

$$l(x_1^2 + x_2^2 + x_3^2) + m(x_1y_1 + x_2y_2 + x_3y_3) + n(x_1z_1 + x_2z_2 + x_3z_3) = p(x_1 + x_2 + x_3)$$

$$\text{or } l.a^2 + m.0 + n.0 = p(x_1 + x_2 + x_3),$$

[using the relations (4) and (5) of 8.18]

$$\text{or } l = p(x_1 + x_2 + x_3)/a^2.$$

Example 3. Prove that the pole of the plane PQR lies on the ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 3.$$

(Gorakhpur 2004, 08, 13, 16)

OR

Prove that the locus of the pole of the plane PQR is

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 3.$$

Sol. Proceeding as in Problem 2 above the equation of the plane PQR is

$$(x/a^2)(x_1 + x_2 + x_3) + (y/b^2)(y_1 + y_2 + y_3) + (z/c^2)(z_1 + z_2 + z_3) = 1 \quad \dots(1)$$

Let the pole of the plane (1) with respect to the given ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ be (α, β, γ) .

The equation of the polar plane of (α, β, γ) with respect to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ is

$$\alpha x/a^2 + \beta y/b^2 + \gamma z/c^2 = 1 \quad \dots(2)$$

Comparing (1) and (2), we get

$$\frac{\alpha}{x_1 + x_2 + x_3} = \frac{\beta}{y_1 + y_2 + y_3} = \frac{\gamma}{z_1 + z_2 + z_3} = \frac{1}{1}.$$

$$\therefore \alpha = x_1 + x_2 + x_3, \beta = y_1 + y_2 + y_3, \gamma = z_1 + z_2 + z_3.$$

Now, we have

$$\begin{aligned} \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} &= \frac{(x_1 + x_2 + x_3)^2}{a^2} + \frac{(y_1 + y_2 + y_3)^2}{b^2} + \frac{(z_1 + z_2 + z_3)^2}{c^2} \\ &= \frac{\Sigma x_1^2}{a^2} + \frac{\Sigma y_1^2}{b^2} + \frac{\Sigma z_1^2}{c^2} + 2 \left\{ \left(\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} \right) \right. \\ &\quad \left. + \left(\frac{x_2 x_3}{a^2} + \frac{y_2 y_3}{b^2} + \frac{z_2 z_3}{c^2} \right) + \left(\frac{x_3 x_1}{a^2} + \frac{y_3 y_1}{b^2} + \frac{z_3 z_1}{c^2} \right) \right\} \\ &= \frac{a^2}{a^2} + \frac{b^2}{b^2} + \frac{c^2}{c^2} \text{ [Using relations (4) and (3) of 8.18 above]} \\ &= 3. \end{aligned}$$

\therefore The locus of the pole (α, β, γ) is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 3$.