9
Jacobians

1. Jacobian of Functions of Two Independent Variables

Definition : *If u and v be the differentiable functions of two independent variables x and y, then the determinant*

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u and v with respect to x and y. It is denoted by $\dfrac{\partial(u, v)}{\partial(x, y)}$ *or* $J(u, v)$.

2. Jacobian of Functions of Three Independent Variables

Definition : *If u, v and w be the differentiable functions of three independent variables x, y and z, then the determinant*

$$\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

is called the Jacobian of u, v and w with respect to x, y and z. It is denoted by $\dfrac{\partial(u, v, w)}{\partial(x, y, z)}$ *or* $J(u, v, w)$.

3. Jacobian of Functions of *n* Independent Variables

Definition : *If* $u_1, u_2, u_3, \ldots, u_n$ *be the differentiable functions of n independent variables* $x_1, x_2, \ldots, x_n$, *then the determinant*

$$\begin{vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_1}{\partial x_3} & \cdots & \dfrac{\partial u_1}{\partial x_n} \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_2}{\partial x_3} & \cdots & \dfrac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial u_n}{\partial x_1} & \dfrac{\partial u_n}{\partial x_2} & \dfrac{\partial u_n}{\partial x_3} & \cdots & \dfrac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called Jacobian of $u_1, u_2, ..., u_n$ with respect to the variables $x_1, x_2, x_3, ..., x_n$

It is denoted by

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} \text{ or } J(u_1, u_2, ..., u_n).$$

4. Some Theorems on Jacobians

Theorem 1. *If $u_1, u_2, ..., u_n$ are functions of $x_1, x_2, ..., x_n$ are functions of $x_1, x_2, ..., x_n$ of the form*

$$u_1 = f_1(x_1)$$
$$u_2 = f_2(x_1, x_2)$$
$$\cdots \quad \cdots$$
$$u_n = f_n(x_1, x_2, ..., x_n)$$

then $$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

Proof : We know that

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ...; x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} & .... & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} & .... & \frac{\partial u_2}{\partial x_n} \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & .... & \frac{\partial u_n}{\partial x_n} \end{vmatrix} \quad ...(1)$$

Since u_1 is a function of x_1 only, therefore ,

$$\frac{\partial u_1}{\partial x_2} = 0, \ \frac{\partial u_1}{\partial x_3} = 0, ... \ \frac{\partial u_1}{\partial x_n} = 0.$$

Also u_2 is a function of x_1 and x_2, hence only $\frac{\partial u_2}{\partial x_1}$ and $\frac{\partial u_2}{\partial x_2}$ will exist. The rest will be zero,

i.e., $$\frac{\partial u_2}{\partial x_3} = 0, \ \frac{\partial u_2}{\partial x_4} = 0, ..., \ \frac{\partial u_2}{\partial x_n} = 0.$$

Again, since u_3 is a function of x_1, x_2 and x_3, therefore $\frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3}$ will exist and the rest will be zero.

Similarly, u_n is a function of $x_1, x_2, ..., x_n$ therefore, all $\frac{\partial u_n}{\partial x_1}, \frac{\partial u_n}{\partial x_2}, ..., \frac{\partial u_n}{\partial x_n}$ will exist.

Putting these values in (1), we get

$$\frac{\partial(u_1, u_2, u_3, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & 0 & 0 & .... & 0 \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & 0 & .... & 0 \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} & .... & 0 \\ .... & .... & .... & .... & .... \\ .... & .... & .... & .... & .... \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \frac{\partial u_n}{\partial x_3} & .... & \frac{\partial u_n}{\partial x_n} \end{vmatrix}.$$

Expanding the determinant in terms of first row, we get

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *If $x = u(1 + v)$ and $y = v(1 + u)$, then find the value of $\frac{\partial(x, y)}{\partial(u, v)}$.*

Solution : We have given $x = u(1 + v)$ and $y = v(1 + u)$

$$\frac{\partial x}{\partial u} = 1 + v; \qquad \frac{\partial x}{\partial v} = u; \frac{\partial y}{\partial u} = v; \frac{\partial y}{\partial v} = 1 + u.$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} (1 + v) & u \\ v & (1 + u) \end{vmatrix}$$

$$= (1 + u)(1 + v) - uv = 1 + u + v + uv - uv$$

$$= 1 + u + v.$$

Example 2. *If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$ then show that*

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2 \sin\theta. \qquad \textbf{(Purvanchal 2002, 05)}$$

Solution : Here given that

$$x = r\sin\theta\cos\phi, \; y = r\sin\theta\sin\phi, \; z = r\cos\theta$$

$$\therefore \quad \frac{\partial x}{\partial r} = \sin\theta\cos\phi$$

$$\frac{\partial x}{\partial \theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi,\ \frac{\partial y}{\partial r} = \sin\theta\sin\phi$$

$$\frac{\partial y}{\partial \theta} = -r\cos\theta\sin\phi,\ \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$$

and $$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial \theta} = -r\sin\theta,\ \frac{\partial z}{\partial \phi} = 0.$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\sin\theta \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & \cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r\sin\theta\{\sin\theta\cos\phi\,(r\sin\theta\cos\phi)$$
$$-r\cos\theta\cos\phi\,(-\cos\phi\cos\theta)$$
$$-\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)\}$$
$$= r\sin\theta\{r\sin^2\theta\cos^2\phi + r\cos^2\theta\cos^2\phi$$
$$+ r\sin^2\phi\,(\sin^2\theta + \cos^2\theta)\}$$
$$= r\sin\theta\{r\cos^2\phi\,(\sin^2\theta + \cos^2\theta) + r\sin^2\phi\}$$
$$= r^2\sin\theta\{\cos^2\phi + \sin^2\phi\}$$
$$= r^2\sin\theta.$$

Example 3. *If* $x = a\cosh\alpha\cos\beta, y = a\sinh\alpha\sin\beta$ *then show that* $\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \frac{a^2}{2}[\cosh 2\alpha - \cos 2\beta]$.

(Agra 2005; Rohilkhand 2004)

Solution : $$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{vmatrix} \quad \text{...(i)}$$

Differentiate partially with respect to α and β from the equations

$$x = a\cosh\alpha\cos\beta$$

$$y = a\sinh\alpha\sin\beta, \quad \text{...(ii)}$$

and ...(iii)

We get

$$\frac{\partial x}{\partial\alpha} = a\sinh\alpha\cos\beta, \quad \frac{\partial x}{\partial\beta} = -a\cosh\alpha\sin\beta$$

$$\frac{\partial y}{\partial\alpha} = a\cosh\alpha\sin\beta \text{ and } \frac{\partial y}{\partial\beta} = a\sinh\alpha\cos\beta$$

Putting in (i)

$$\frac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} a\sinh\alpha\cos\beta & -a\cosh\alpha\sin\beta \\ a\cosh\alpha\sin\beta & a\sinh\alpha\cos\beta \end{vmatrix}$$

$$= a^2(\sinh^2\alpha\cos^2\beta + \cosh^2\alpha\sin^2\beta)$$

$$= a^2[(\cosh^2\alpha - 1)\cos^2\beta + \cosh^2\alpha(1-\cos^2\beta)]$$

$$= a^2[\cosh^2\alpha\cos^2\beta - \cos^2\beta + \cosh^2\alpha - \cosh^2\alpha\cos^2\beta]$$

$$= a^2[\cosh^2\alpha - \cos^2\beta]$$

$$= \frac{a^2}{2}[\cosh 2\alpha + 1 - 1 - \cos 2\beta]$$

$$= \frac{a^2}{2}[\cosh 2\alpha - \cos 2\beta].$$

Example 4. *Find the jacobian* $J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$, *given that* $x = r\cos\theta\cos\phi$, $y = r\sin\theta\sqrt{1-m^2\sin^2\phi}$, $z = r\sin\phi\sqrt{1-n^2\sin^2\theta}$, *where* $m^2+n^2=1$.

Solution : Squaring the given equation and adding, we have

$$x^2+y^2+z^2 = r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta(1-m^2\sin^2\phi) + r^2\sin^2\phi(1-n^2\sin^2\theta)$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta + r^2\sin^2\phi - r^2(m^2+n^2)\sin^2\theta\sin^2\phi$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta + r^2\sin^2\phi - r^2\sin^2\theta\sin^2\phi$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta + r^2\sin^2\phi(1-\sin^2\theta)$$

$$= r^2\cos^2\theta\cos^2\phi + r^2\sin^2\theta + r^2\sin^2\phi\cos^2\theta$$

$$= r^2\cos^2\theta(\cos^2\phi + \sin^2\phi) + r^2\sin^2\theta$$

$$= r^2\cos^2\theta + r^2\sin^2\theta$$

$$= r^2(\cos^2\theta + \sin^2\theta) = r^2$$

i.e., $x^2+y^2+z^2 = r^2$.

Differentiating the equation $x^2 + y^2 + z^2 = r^2$ w.r.t. r, θ, ϕ respectively it is clear that

$$\left.\begin{aligned} x\frac{\partial x}{\partial r} + y\frac{\partial y}{\partial r} + z\frac{\partial z}{\partial r} &= r \\ x\frac{\partial x}{\partial \theta} + y\frac{\partial y}{\partial \theta} + z\frac{\partial z}{\partial \theta} &= 0 \\ x\frac{\partial x}{\partial \phi} + y\frac{\partial y}{\partial \phi} + z\frac{\partial z}{\partial \phi} &= 0 \end{aligned}\right\} \qquad \text{...(A)}$$

Now, $$J = \frac{\partial (x, y, z)}{\partial (r, \theta, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \frac{1}{xyz}\begin{vmatrix} x\frac{\partial x}{\partial r} & x\frac{\partial x}{\partial \theta} & x\frac{\partial x}{\partial \phi} \\ y\frac{\partial y}{\partial r} & y\frac{\partial y}{\partial \theta} & y\frac{\partial y}{\partial \phi} \\ z\frac{\partial z}{\partial r} & z\frac{\partial z}{\partial \theta} & z\frac{\partial z}{\partial \phi} \end{vmatrix}$$

By transformation $R_1 \to R_1 + R_2 + R_3$, we have

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \frac{1}{xyz}\begin{vmatrix} r & 0 & 0 \\ y\frac{\partial y}{\partial r} & y\frac{\partial y}{\partial \theta} & y\frac{\partial y}{\partial \phi} \\ z\frac{\partial z}{\partial r} & z\frac{\partial z}{\partial \theta} & z\frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \frac{r}{x}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \phi} - \frac{\partial z}{\partial \theta}\cdot\frac{\partial y}{\partial \phi}\right)$$

$$= \frac{r}{x}\left[r\cos\theta\sqrt{(1 - m^2\sin^2\phi)}\, r\cos\phi\sqrt{(1 - n^2\sin^2\theta} - \frac{r\sin\theta\, m^2\sin\phi\cos\phi}{\sqrt{1 - m^2\sin^2\phi}}\cdot\frac{r\sin\phi\, n^2\sin\theta\cos\theta}{\sqrt{1 - n^2\sin^2\theta}}\right]$$

$$= \frac{r^3\cos\theta\cos\phi\,[(1 - m^2\sin^2\phi)(1 - n^2\sin^2\theta) - m^2n^2\sin^2\theta\sin^2\phi]}{r\cos\theta\cos\phi\sqrt{(1 - m^2\sin^2\phi)}\sqrt{(1 - n^2\sin^2\theta)}}$$

$$= \frac{r^2(1 - m^2\sin^2\phi - n^2\sin^2\theta)}{\sqrt{(1 - m^2\sin^2\phi)(1 - n^2\sin^2\theta)}}.$$

Example 5. *If $x = r\cos\theta, y = r\sin\theta$, then find the values of $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$.* **(Agra 2006)**

Solution : We have given

$$x = r\cos\theta, y = r\sin\theta$$

Now
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

Since $x = r\cos\theta, y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{r^2} = \frac{-\sin\theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}.$$

Now
$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{vmatrix}$$

EXERCISE 9 (A)

1. If $u = ax + by, v = cx + dy$, then find the value of $J = \frac{\partial(u,v)}{\partial(x,y)}$.
2. If $y_1 = \sin x_1$, $y_2 = \cos x_1 \sin x_2$, and $y_3 = \cos x_1, \cos x_2, \sin x_3$, then show t hat
$$J(y_1, y_2, y_3) = \cos^3 x_1 \cos^2 x_2 \cos x_3.$$
3. If $x_1 = \cos\theta_1, x_2 = \sin\theta_1 \cos\theta_2, x_3 = \sin\theta_1 \sin\theta_2 \cos\theta_3, ..., x_n = \sin\theta_1 \sin\theta_2 ... \cos\theta_n$, then prove that
$$\frac{\partial(x_1, x_2, x_3, ..., x_n)}{\partial(\theta_1, \theta_2, \theta_3, ..., \theta_n)} = (-1)^n \sin^n\theta_1 \sin^{n-1}\theta_2 ... \sin\theta_n.$$
4. If $x = r\cos\theta\cos\phi, y = r\cos\theta\sin\phi, z = r\sin\theta$, then show that
$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = -r^2\cos\theta.$$

5. If $x = \sin\theta\sqrt{1 - c^2\sin^2\phi}, y = \cos\theta\cos\phi$, then prove that

$$\frac{\partial(x,y)}{\partial(\theta,\phi)} = -\sin\phi\left[\frac{(1-c^2)\cos^2\theta + c^2\cos^2\phi}{\sqrt{1-c^2\sin^2\phi}}\right].$$

6. If $y_1 = r\sin\theta_1\sin\theta_2$, $y_2 = r\sin\theta_1\cos\theta_2$, $y_3 = r\cos\theta_1\sin\theta_3$, $y_4 = r\cos\theta_1\cos\theta_3$, then find the value of

$$J = \frac{\partial(y_1, y_2, y_3, y_4)}{\partial(r, \theta_1, \theta_2, \theta_3)}.$$

[Hint : $y_1^2 + y_2^2 + y_3^2 + y_4^2 = r^2$, and solved parallel to example 7]

7. If $u = 3x + 2y - z$; $v = x - 2y + z$; $w = x(x + 2y - z)$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$.

8. If $u = \frac{y^2}{2x}, v = \frac{x^2 + y^2}{2x}$, then find the value of $\frac{\partial(u,v)}{\partial(x,y)}$.

9. If $y_1 = \frac{x_2x_3}{x_1}, y_2 = \frac{x_3x_1}{x_2}, y_3 = \frac{x_1x_2}{x_3}$, prove that

$$J(y_1, y_2, y_3) = 4.$$ (Agra 2008; Rohilkhand 2007)

10. If $x = u(1-u)$ and $y = v(1-u)$, find the Jacobian of x, y with respect to u, v. (Kanpur 2006)

11. If $u = 1 - x, v = x(1-y)$ and $w = xy(1-z)$, prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -x^2y.$$ (Kanpur 2007)

ANSWERS

1. $ad - bc$ 6. $J = r^2\sin\theta_1\cos\theta_1$ 8. $-y/2x$ 10. $1 + u + v$

5. Jacobian of Function of Functions

Theorem 2. *If u_1, u_2 are functions of y_1, y_2 and y_1, y_2 are functions of x_1, x_2 then*

$$\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \frac{\partial(u_1,u_2)}{\partial(y_1,y_2)}\cdot\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}.$$

Proof :

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2}$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1}$$

$$\frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2}$$

Now $$\frac{\partial(u_1, u_2)}{\partial(y_1, y_2)}\cdot\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} & \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_2}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} \\ \frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} & \frac{\partial u_2}{\partial y_1}\cdot\frac{\partial y_2}{\partial x_2} + \frac{\partial u_2}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{vmatrix} = \frac{\partial(u_1, u_2)}{\partial(x_1, x_2)}.$$

Theorem 3. *If u_1, u_2, u_3 are functions of y_1, y_2, y_3, and y_1, y_2, y_3 are functions of x_1, x_2, x_3 then*

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \frac{\partial(u_1, u_2, u_3)}{\partial(y_1, y_2, y_3)} \times \frac{\partial(y_1, y_2, y_3)}{\partial(x_1, x_2, x_3)}.$$

(Kumaun 2003)

Proof : u_1, u_2 and u_3 are functions of y_1, y_2 and y_3. Also y_1, y_2, y_3 are functions of x_1, x_2 and x_3.

Hence, we get

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_1} + \frac{\partial u_1}{\partial y_3}\cdot\frac{\partial y_3}{\partial x_1}$$

$$= \sum_{i=1}^{3} \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_2} + \frac{\partial u_1}{\partial y_2}\cdot\frac{\partial y_2}{\partial x_2} + \frac{\partial u_1}{\partial y_3}\cdot\frac{\partial y_3}{\partial x_3}$$

$$= \sum \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2}.$$

Similarly, $$\frac{\partial u_1}{\partial x_3} = \sum \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_3}$$

$$\frac{\partial u_2}{\partial x_1} = \sum \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1}, \frac{\partial u_2}{\partial x_2} = \sum \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2}, \frac{\partial u_2}{\partial x_3} = \sum \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_3}$$

$$\frac{\partial u_3}{\partial x_1} = \sum \frac{\partial u_3}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1}, \frac{\partial u_3}{\partial x_2} = \sum \frac{\partial u_3}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} \text{ and } \frac{\partial u_3}{\partial x_3} = \sum \frac{\partial u_3}{\partial y_3}\cdot\frac{\partial y_3}{\partial x_3}.$$

Now consider

$$\frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)} \times \frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \frac{\partial u_1}{\partial y_3} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \frac{\partial u_2}{\partial y_3} \\ \frac{\partial u_3}{\partial y_1} & \frac{\partial u_3}{\partial y_2} & \frac{\partial u_3}{\partial y_3} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_3} \\ \Sigma \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_3} \\ \Sigma \frac{\partial u_3}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1} & \Sigma \frac{\partial u_3}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \Sigma \frac{\partial u_3}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_3} \end{vmatrix}$$

Putting the values of each element of the determinant from the above relations, we get

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}.$$

$$\therefore \quad \frac{\partial(u_1,u_2,u_3)}{\partial(y_1,y_2,y_3)}\cdot\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}.$$

Generalization :

Theorem 4. *If* $u_1, u_2, u_3, ..., u_n$ *are functions of* $y_1, y_2, y_3, ..., y_n$ *and* $y_1, y_2, y_3, ..., y_n$ *are functions of* $x_1, x_2, x_3, ..., x_n$, *then*

$$\frac{\partial(u_1,u_2,\ldots u_n)}{\partial(x_1,x_2,\ldots,x_n)} = \frac{\partial(u_1,u_2,u_3,\ldots,u_n)}{\partial(y_1,y_2,\ldots,y_n)}\,\frac{\partial(y_1,y_2,\ldots,y_n)}{\partial(x_1,x_2,\ldots,x_n)}.$$

Proof : By hypothesis, variables $u_1, u_2, ..., u_n$ are functions of variables $y_1, y_2, ..., y_n$ which are themselves functions of variables $x_1, x_2, ..., x_n$.

Therefore,
$$\frac{\partial u_j}{\partial x_i} = \frac{\partial u_j}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \frac{\partial u_j}{\partial y_2}\frac{\partial y_2}{\partial x_i} + \ldots + \frac{\partial u_j}{\partial y_n}\frac{\partial y_n}{\partial x_i}$$
$$= \sum_{r=1}^{n} \frac{\partial u_j}{\partial y_r}\frac{\partial y_r}{\partial x_i}. \quad \ldots(1)$$

Thus,
$$\frac{\partial u_1}{\partial x_i} = \sum_{r=1}^{n} \frac{\partial u_1}{\partial y_r}\frac{\partial y_r}{\partial x_i}, \quad \frac{\partial u_2}{\partial x_i} = \sum_{r=1}^{n} \frac{\partial u_2}{\partial y_r}\frac{\partial y_r}{\partial x_i} \ldots \text{ and so on.}$$

Now, by 'row by column rule' for multiplication of determinants,

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(y_1, y_2, ..., y_n)} \cdot \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial y_1} & \frac{\partial u_1}{\partial y_2} & \cdots & \frac{\partial u_1}{\partial y_n} \\ \frac{\partial u_2}{\partial y_1} & \frac{\partial u_2}{\partial y_2} & \cdots & \frac{\partial u_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial y_1} & \frac{\partial u_n}{\partial y_2} & \cdots & \frac{\partial u_n}{\partial y_n} \end{vmatrix} \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma \frac{\partial u_1}{\partial y_r} \frac{\partial y_r}{\partial x_1} & \Sigma \frac{\partial u_1}{\partial y_r} \frac{\partial y_r}{\partial x_2} & \cdots & \Sigma \frac{\partial u_1}{\partial y_r} \frac{\partial y_r}{\partial x_n} \\ \Sigma \frac{\partial u_2}{\partial y_r} \frac{\partial y_r}{\partial x_1} & \Sigma \frac{\partial u_2}{\partial y_r} \frac{\partial y_r}{\partial x_2} & \cdots & \Sigma \frac{\partial u_2}{\partial y_r} \frac{\partial y_r}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma \frac{\partial u_n}{\partial y_r} \frac{\partial y_r}{\partial x_1} & \Sigma \frac{\partial u_n}{\partial y_r} \frac{\partial y_r}{\partial x_2} & \cdots & \Sigma \frac{\partial u_n}{\partial y_r} \frac{\partial y_r}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix} = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)}.$$

6. Jacobian of Implicit Functions

Theorem 5. *If u_1, u_2 and u_3 be the functions of independent variables x_1, x_2 and x_3 given by the implicit relations*

$$F_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$
$$F_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$$

then

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}.$$

Proof : Differentiating F_1, F_2, F_3 with respect to x_1, x_2, and x_3, we get

$$\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial u_1} \cdot \frac{\partial u_1}{\partial x_1} + \frac{\partial F_1}{\partial u_2} \cdot \frac{\partial u_2}{\partial x_1} + \frac{\partial F_1}{\partial u_3} \cdot \frac{\partial u_3}{\partial x_1} = 0$$

$$\Rightarrow \quad \sum_{r=1}^{3} \frac{\partial F_1}{\partial u_r} \cdot \frac{\partial u_r}{\partial x_1} = -\frac{\partial F_1}{\partial x_1}$$

$$\frac{\partial F_1}{\partial x_2}+\frac{\partial F_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_2}+\frac{\partial F_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_2}+\frac{\partial F_1}{\partial u_3}\cdot\frac{\partial u_3}{\partial x_3}=0$$

$$\Rightarrow \quad \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2}=-\frac{\partial F_1}{\partial x_2}$$

$$\frac{\partial F_1}{\partial x_3}+\frac{\partial F_1}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_3}+\frac{\partial F_1}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_3}+\frac{\partial F_1}{\partial u_3}\cdot\frac{\partial u_3}{\partial x_3}=0$$

$$\Rightarrow \quad \sum_{r=1}^{3}\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_1}{\partial x_3}.$$

Similarly,

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1}=-\frac{\partial F_2}{\partial x_1},$$

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2}=-\frac{\partial F_2}{\partial x_2}$$

$$\sum_{r=1}^{3}\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_2}{\partial x_3},$$

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1}=-\frac{\partial F_3}{\partial x_1}$$

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2}=-\frac{\partial F_3}{\partial x_2}$$

and

$$\sum_{r=1}^{3}\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}=-\frac{\partial F_3}{\partial x_3}.$$

Now consider,

$$\frac{\partial(F_1,F_2,F_3)}{\partial(u_1,u_2,u_3)}\times\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}=\begin{vmatrix}\frac{\partial F_1}{\partial u_1} & \frac{\partial F_1}{\partial u_2} & \frac{\partial F_1}{\partial u_3}\\ \frac{\partial F_2}{\partial u_1} & \frac{\partial F_2}{\partial u_2} & \frac{\partial F_2}{\partial u_3}\\ \frac{\partial F_3}{\partial u_1} & \frac{\partial F_3}{\partial u_2} & \frac{\partial F_3}{\partial u_3}\end{vmatrix}\times\begin{vmatrix}\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3}\\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3}\\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3}\end{vmatrix}$$

$$=\begin{vmatrix}\Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_1}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\\ \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_2}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\\ \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_1} & \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_2} & \Sigma\frac{\partial F_3}{\partial u_r}\cdot\frac{\partial u_r}{\partial x_3}\end{vmatrix}$$

Putting the values of each summation, we get

$$= \begin{vmatrix} -\frac{\partial F_1}{\partial x_1} & -\frac{\partial F_1}{\partial x_2} & -\frac{\partial F_1}{\partial x_3} \\ -\frac{\partial F_2}{\partial x_1} & -\frac{\partial F_2}{\partial x_2} & -\frac{\partial F_2}{\partial x_3} \\ -\frac{\partial F_3}{\partial x_1} & -\frac{\partial F_3}{\partial x_2} & -\frac{\partial F_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_3}{\partial x_1} & \frac{\partial F_3}{\partial x_2} & \frac{\partial F_3}{\partial x_3} \end{vmatrix}$$

$$= (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)}.$$

$$\therefore \quad \frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = (-1)^3 \frac{\partial(F_1, F_2, F_3)}{\partial(x_1, x_2, x_3)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(u_1, u_2, u_3)}.$$

Theorem 6. (Generalizations) : *If $u_1, u_2, ..., u_n$ be the functions of n independent variables $x_1, x_2, ..., x_n$, given by the implicit relations*

$$F_1 (u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0$$

$$F_2 (u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$F_n(u_1, u_2, ..., u_n, x_1, x_2, ..., x_n) = 0,$$

then

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(x_1, x_2, ..., x_n)}}{\dfrac{\partial(F_1, F_2, ..., F_n)}{\partial(u_1, u_2, ..., u_n)}}.$$

Proof : The proof of this theorem is beyond the scope of this book.

7. Corollary

If $u_1, u_2, ..., u_n$ be the functions of n independent variables $x_1, x_2, ..., x_n$, given by implicit relations of the form

$$F_1 (u_1, x_1, x_2, ..., x_n) = 0$$

$$F_2 (u_1, u_2, x_2, x_3, ..., x_n) = 0$$

$$F_3(u_1, u_2, u_3, x_3, x_4, \ldots, x_n) = 0$$

$$\ldots \quad \ldots \quad \ldots$$

$$F_n(u_1, u_2, \ldots, u_n, x_n) = 0,$$

then
$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial u_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdots \dfrac{\partial F_n}{\partial u_n}}.$$

Proof : Since, by hypothesis

$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial F_1}{\partial x_1} \cdot \frac{\partial F_2}{\partial x_2} \cdots \frac{\partial F_n}{\partial x_n}$$

and
$$\frac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)} = \frac{\partial F_1}{\partial u_1} \frac{\partial F_2}{\partial u_2} \cdots \frac{\partial F_n}{\partial u_n}$$

$\therefore$
$$\frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(x_1, x_2, \ldots, x_n)}}{\dfrac{\partial(F_1, F_2, \ldots, F_n)}{\partial(u_1, u_2, \ldots, u_n)}}$$

$$= (-1)^n \frac{\dfrac{\partial F_1}{\partial x_1} \cdot \dfrac{\partial F_2}{\partial x_2} \cdots \dfrac{\partial F_n}{\partial x_n}}{\dfrac{\partial F_1}{\partial u_1} \cdot \dfrac{\partial F_2}{\partial u_2} \cdots \dfrac{\partial F_n}{\partial u_n}}.$$

ILLUSTRATIVE EXAMPLES

Example 1. *If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$, then prove that*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{y^2 - x^2}{2uv(u - v)}.$$

(Agra 2011; Bundelkand 2006; Kumayun 2002)

Solution : Here $f_1 \equiv u^3 + v^3 - (x + y) = 0$

and $f_2 \equiv u^2 + v^2 - (x^3 + y^3) = 0$

$$\frac{\partial(u, v)}{\partial(x, y)} = (-1)^2 \frac{\dfrac{\partial(f_1, f_2)}{\partial(x, y)}}{\dfrac{\partial(f_1, f_2)}{\partial(u, v)}} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix} \Bigg/ \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= \frac{3(y^2 - x^2)}{6uv(u - v)} = \frac{y^2 - x^2}{2uv(u - v)}.$$

Example 2. *If* $u^3 + v^3 + w^3 = x + y + z$, $u^2 + v^2 + w^2 = x^3 + y^3 + z^3$ *and* $u + v + w = x^2 + y^2 + z^2$ *then show that*

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

Solution : Let
$$F_1 \equiv (u^3 + v^3 + w^3 - x - y - z),$$
$$F_2 \equiv (u^2 + v^2 + w^2 - x^3 - y^3 - z^3)$$
and
$$F_3 \equiv (u + v + w - x^2 - y^2 - z^2)$$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^3 \frac{\dfrac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)}}{\dfrac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)}} \quad ...(i)$$

Now consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{vmatrix} \quad ...(ii)$$

From F_1, F_2 and F_3, we get

$$\frac{\partial F_1}{\partial x} = -1, \frac{\partial F_1}{\partial y} = -1, \frac{\partial F_1}{\partial z} = -1$$

$$\frac{\partial F_2}{\partial x} = -3x^2, \frac{\partial F_2}{\partial y} = -3y^2, \frac{\partial F_2}{\partial z} = -3z^2$$

$$\frac{\partial F_3}{\partial x} = -2x, \frac{\partial F_3}{\partial y} = -2y, \frac{\partial F_3}{\partial z} = -2z.$$

Putting these values in (ii), we get

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & -1 \\ 3(z^2 - x^2) & 3(z^2 - y^2) & -3z^2 \\ 2(z - x) & 2(z - y) & -2z \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & -1 \\ (z-x)(z+x) & (z-y)(z+y) & -3z^2 \\ (z-x) & z-y & -2z \end{vmatrix}$$

$$= 6(z-x)(z-y)\begin{vmatrix} 0 & 0 & -1 \\ z+x & z+y & -3z^2 \\ 1 & 1 & -2z \end{vmatrix}$$

$$= 6(z-x)(z-y)(-1)\begin{vmatrix} z+x & z+y \\ 1 & 1 \end{vmatrix}$$

$$= -6(z-x)(z-y)(x-y)$$

$$= 6(x-y)(y-z)(z-x).$$

Now consider

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} \quad \text{...(iii)}$$

Again from F_1, F_2, and F_3, we get

$$\frac{\partial F_1}{\partial u} = 3u^2, \frac{\partial F_1}{\partial v} = 3v^2, \frac{\partial F_1}{\partial w} = 3w^2$$

$$\frac{\partial F_2}{\partial u} = 2u, \frac{\partial F_2}{\partial v} = 2v, \frac{\partial F_2}{\partial w} = 2w$$

$$\frac{\partial F_3}{\partial u} = 1, \frac{\partial F_3}{\partial v} = 1, \frac{\partial F_3}{\partial w} = 1$$

Putting in (iii),

$$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix}.$$

$C_1 \to C_1 - C_3$, $C_2 \to C_2 - C_3$, gives us

$$= 6\begin{vmatrix} u^2 - w^2 & v^2 - w^2 & 3w^2 \\ u - w & v - w & 2w \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6(u-w)(v-w)\begin{vmatrix} u+w & v+w & 3w^2 \\ 1 & 1 & 2w \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6(u-w)(v-w)\begin{vmatrix} u+w & v+w \\ 1 & 1 \end{vmatrix}$$

$$= 6(u-w)(v-w)(u-v)$$

$$= -6(u-v)(v-w)(w-u)$$

$$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{6(x-y)(y-z)(z-x)}{(-6)(u-v)(v-w)(w-u)}$$

$$= (-1)\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

$$= \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}.$$

Example 3. *If u, v, w are the roots of the cubic*

$$(\lambda-x)^3 + (\lambda-v)^3 + (\lambda-z)^3 = 0$$

in λ, find ∂ (u, v, w)/∂(x, y, z).

Solution : If u, v, w be the roots of the equation

$$(\lambda-x)^3 + (\lambda-y)^3 + (\lambda-z)^3 = 0$$

$$3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$$

then,

$$S_1 = u+v+w = \frac{3(x+y+z)}{3}$$

$$S_2 = uv+vw+wu = \frac{3(x^2+y^2+z^2)}{3}$$

$$S_3 = uvw = \frac{x^3+y^3+z^3}{3}.$$

Let,

$$F_1 \equiv u+v+w-x-y-z = 0$$

$$F_2 \equiv uv+vw+wu-(x^2+y^2+z^2) = 0$$

$$F_3 \equiv uvw - \tfrac{1}{3}(x^3+y^3+z^3) = 0$$

$$\therefore \quad \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)}}{\dfrac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)}} \quad \text{...(i)}$$

$$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z} \\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_2}{\partial z} \\ \dfrac{\partial F_3}{\partial x} & \dfrac{\partial F_3}{\partial y} & \dfrac{\partial F_3}{\partial z} \end{vmatrix}.$$

Putting the values, we get

$$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= (-2)\begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= (-2)\begin{vmatrix} y-x & z-x \\ (y-x)(y+x) & (z-x)(z+x) \end{vmatrix}$$

$$= -2(y-x)(z-x)[z+x-y-x]$$

$$= -2(x-y)(y-z)(z-x)$$

Now, $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & uw & uv \end{vmatrix}$$

$C_2 \to C_2 - C_1, C_3 \to C_3 - C_1, C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix}$$

$$= \begin{vmatrix} u-v & u-w \\ w(u-v) & v(u-w) \end{vmatrix}$$

$$= (u-v)(u-w)\begin{vmatrix} 1 & 1 \\ w & v \end{vmatrix}$$

$$= (u-v)(u-w)(v-w) = -(u-v)(v-w)(w-u)$$

Hence,

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{-2(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-v)}(-1)^3$$

$$J(u, v, w) = \frac{-2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-v)}.$$

Example 4. *If* $u^3 + v + w = x + y^2 + z^2$

$$u + v^3 + w = x^2 + y + z^2$$

$$u + v + w^3 = x^2 + y^2 + z.$$

then prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(xy+yz+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$

Solution : Let $f_1 = u^3 + v + w - x - y^2 - z^2$

$f_2 = u + v^3 + w - x^2 - y - z^2$

$f_3 = u + v + w^3 - x^2 - y^2 - z.$

Now $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} \quad \text{...(i)}$$

$$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -2y & -2z \\ -2x & -1 & -2z \\ -2x & -2y & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2y & 2z \\ 2x & 1 & 2z \\ 2x & 2y & 1 \end{vmatrix}$$

$$= 1(1-4yz) - 2y(2x-4xz) + 2z(4xy-2x)$$

$$= 1 - 4(yz+xy+zx) + 16xyz \quad \text{...(ii)}$$

and $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 1 & 1 \\ 1 & 3v^2 & 1 \\ 1 & 1 & 3w^2 \end{vmatrix}$$

$$= 3u^2(9v^2w^2 - 1) - 1(3w^2 - 1) + 1(1 - 3v^2)$$

$$= 2 - 3(u^2+v^2+w^2) + 27u^2v^2w^2. \quad \text{...(iii)}$$

From equations (i), (ii) and (iii), we have

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1-4(yz+xy+zx)+16xyz}{2-3(u^2+v^2+w^2)+27u^2v^2w^2}.$$

Theorem 5. *If the Jacobians of u_1 and u_2 w.r.t. x_1, x_2 is $J = J_1$ and Jacobians of x_1 and x_2 w.r.t., u_1, u_2 is $J = J_2$, then prove that* **(Agra 2007, 10)**

$$J_1 \cdot J_2 = 1.$$

Theorem 7. *Prove that* $\dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \dfrac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = 1.$

Proof : Here $\dfrac{\partial(x_1, x_2, \ldots, x_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \dfrac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, y_n)}$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n} \end{vmatrix} \times \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \sum_{i=1}^{n} \frac{\partial x_i}{\partial y_i}\cdot\frac{\partial y_1}{\partial x_1} & \sum_{i=1}^{n} \frac{\partial x_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \cdots & \sum_{i=1}^{n} \frac{\partial x_1}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_n} \\ \sum_{i=1}^{n} \frac{\partial x_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1} & \sum_{i=1}^{n} \frac{\partial x_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \cdots & \sum_{i=1}^{n} \frac{\partial x_2}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \sum_{i=1}^{n} \frac{\partial x_n}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_1} & \sum_{i=1}^{n} \frac{\partial x_n}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_2} & \cdots & \sum_{i=1}^{n} \frac{\partial x_n}{\partial y_i}\cdot\frac{\partial y_i}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_n} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial x_1} & \frac{\partial x_n}{\partial x_2} & \cdots & \frac{\partial x_n}{\partial x_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{vmatrix} = 1.$$

EXERCISE 9 (B)

1. If $u = ax + by$, $v = cx + dy$, then find the value of $J(u, v)$.
2. If $u = \frac{y^2}{2x}$, $v = \frac{x^2 + y^2}{2x}$, then find the value of $\frac{\partial(u, v)}{\partial(x, y)}$.
3. If $u = \frac{1}{x}$, $v = \frac{x^2}{y}$, $w = x + y + zy^2$, then find the value of $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.
4. If $u = \frac{xy}{1 - xy}$, $v = \tan^{-1} x + \tan^{-1} y$, then find the value of $\frac{\partial(u, v)}{\partial(x, y)}$.

   $\left[\text{Hint}: x = \tan\theta, y = \tan\phi \text{ and } \frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(\theta, \phi)}\cdot\frac{\partial(\theta, \phi)}{\partial(x, y)}\right]$