Q 1. Write down Taylor's theorem for functions of two variables. (2011)

Sol. If $f(x, y)$ and all its partial derivatives upto nth order are finite and continuous for all points (x, y), where $a \leq x \leq a + h$ and $b \leq y \leq b + k$.
Then,

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$+ \ldots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a, b) + R_n$$

where, $R_n = \dfrac{1}{n!}\left(h\dfrac{\partial}{\partial x} + k\dfrac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0 < \theta < 1.$

It can be written as in the following form

$$f(a+h, b+k) = f(a, b) + df(a, b) + \frac{1}{2!} d^2 f(a, b) + \ldots + \frac{1}{(n-1)!} d^{n-1} f(a, b)$$

$$+ \frac{1}{n!} d^n f(a+\theta h, b+\theta k), \text{ where } d = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}$$

Q 2. Find the first six terms of the expansion of the function $e^x \log(1 + y)$ in a Taylor's series in the neighbourhood of the point (0, 0).

Sol. We have,

		$x=0, y=0$
$f(x, y)$	$e^x \log(1+y)$	0
$\frac{\partial f}{\partial x}$	$e^x \log(1+y)$	0
$\frac{\partial f}{\partial y}$	$e^x \frac{1}{1+y}$	1
$\frac{\partial^2 f}{\partial x^2}$	$e^x \log(1+y)$	0
$\frac{\partial^2 f}{\partial y^2}$	$-\frac{e^x}{(1+y)^2}$	-1
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{e^x}{1+y}$	1

Taylor's series is $f(x, y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$

$$+\frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x^2} + y^2\frac{\partial^2 f}{\partial y^2} + 2xy\frac{\partial f}{\partial x\partial y}\right) + \ldots$$

$$\Rightarrow \quad e^x \log(1+y) = 0 + (x\times 0 + y\times 1) + \frac{1}{2}[x^2\times 0 + 2xy\times 1 + y^2(-1)] + \ldots$$

$$= y + xy - \frac{y^2}{2}$$

Q 3. Expand $x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$.
(2016, 15, 09, 07, 05)

Sol. We have,

		$x=1, \ y=-2$
$f(x, y)$	$x^2y + 3y - 2$	-10
$\frac{\partial f}{\partial x}$	$2xy$	-4
$\frac{\partial f}{\partial y}$	$x^2 + 3$	4
$\frac{\partial^2 f}{\partial x^2}$	$2y$	-4
$\frac{\partial^2 f}{\partial x \partial y}$	$2x$	2

$\frac{\partial^2 f}{\partial y^2}$	0	0
$\frac{\partial^3 f}{\partial x^3}$	0	0
$\frac{\partial^3 f}{\partial y^3}$	0	0
$\frac{\partial^3 f}{\partial x^2 \partial y}$	2	2

and all other higher derivatives are zero.

$$\therefore \quad f(\overline{x-a}+a, \overline{y-b}+b) = f(a,b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + (y-b)^2\frac{\partial^2 f}{\partial y^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}\right] + \ldots$$

$$\therefore x^2y+3y-2 = -10-4(x-1)+4(y+2)+\frac{1}{2}[(x-1)^2(-4)+(y+2)^2$$

$$\times 0 + 2(x-1)(y+2)\times 2] + \left(\frac{x-1}{2}\right)^2 (y+2)\times 2$$

$$= -10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2)+(x-1)^2(y+2)$$

Q 4. Expand $y^2x+3x-2$ in powers of $(y-1)$ and $(x+2)$. **(2008)**

Sol. Do same as Q. 3.

Ans. $4x-4y+11+2(x+2)(y-1)-2(y-1)^2+(x+2)(y-1)^2$

Short Answer Questions

Q 1. Expand $x^2y+3y-2$ in power of $(x+1)$ and $(y+2)$. **(2018)**

Sol. We have,

		$x=-1, y=-2$
$f(x,y)$	$x^2y+3y-2$	$(-1)^2(-2)+3(-2)$ $-2=-10$
$\frac{\partial f}{\partial x}$	$2xy$	$2(-1)(-2)=4$

$\frac{\partial f}{\partial y}$	$x^2 + 3$	$(-1)^2 + 3 = 4$
$\frac{\partial^2 f}{\partial x^2}$	$2y$	$2(-2) = -4$
$\frac{\partial^2 f}{\partial x \partial y}$	$2x$	$2(-1) = -2$
$\frac{\partial^2 f}{\partial y^2}$	0	0
$\frac{\partial^3 f}{\partial x^3}$	0	0
$\frac{\partial^3 f}{\partial y^3}$	0	0
$\frac{\partial^3 f}{\partial x^2 \partial y}$	2	2

and all other higher derivatives are zero.

$$\therefore f(\overline{x-a}+a, \overline{y-b}+b) = f(x, b) = \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right]$$

$$+ \frac{1}{\partial !}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + (y-b)^2\frac{\partial^2 f}{\partial y^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y}\right] + \ldots$$

$$\therefore x^2 y + 3y - 2 = -10 + (x+1)(4) + (y+2)(4)$$

$$+ \frac{1}{2}[(x+1)^2(-4) + (y+2)^2 \times 0 + 2(x+1)(y+1)(-2)]$$

$$+ \frac{1}{3!}[3(x+1)^2(y+2)(2)]$$

$$= 10 + 4(x+1) + 4(y+2) - 2(x+1)^2 - 2(x+1)(y+1)$$

$$+ (x+1)^2(y+2)$$

Q 1. State and prove Taylor's theorem for the function of two variables. Expand the function $f(x,y)=\tan^{-1}xy$ in the neighbourhood of the point (1,–1) upto second degree terms using Taylor's theorem. Hence, compute $f(0.9,-1.2)$ approximately. **(2014)**

Sol. Part I Statement If $f(x, y)$ and all its partial derivatives upto nth order are finite and continuous for all points (x, y), where $a \le x \le a+h$ and $b \le y \le b+k$.

Then, $f(a+h, b+k) = f(a, b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\ldots+\frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n-1} f(a,b)+R_n$$

where, $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k),\ 0<\theta<1.$

It can be written as in the following form

$$f(a+h, b+k) = f(a, b) + df(a, b) + \frac{1}{2!}d^2 f(a, b) + \ldots + \frac{1}{(n-1)!}d^{n-1} f(a, b)$$

$$+\frac{1}{n!}d^n f(a+\theta h, b+\theta k), \text{ where } d = h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}.$$

Proof Let $x = a + th$, $y = b + tk$, where $0 \le t \le 1$, is a parameter, then $f(x, y) = f(a+th, b+tk) = \phi(t)$

Since, the partial derivatives of $f(x, y)$ of order n are continuous in the domain under consideration, $\phi^n(t)$ is continuous in [0,1] and also

$$\phi'(t) = \frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} = h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f$$

$$\phi''(t) = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\phi^{(n)}(t) = \left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f$$

Therefore, by Maclaurin's theorem, we have

$$\phi(t) = \phi(0) + t\phi'(0) + \frac{t^2}{2!}\phi''(0) + \ldots + \frac{t^{n-1}}{(n-1)!}\phi^{(n-1)}(0) + \frac{t^n}{n!}\phi^n(0) \text{ where, } 0<\theta<1.$$

Now, putting $t = 1$, we get

$$\phi(1) = \phi(0) + \phi'(0) + \frac{1}{2!}\phi''(0) + \ldots + \frac{1}{(n-1)!}\phi^{n-1}(0) + \frac{1}{n!}\phi^{(n)}(0)$$

But $\quad \phi(1) = f(a+h, b+k)$ and $\phi(0) = f(a, b)$

$$\phi'(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a, b)$$

$$\phi''(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a, b)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\phi^n(0) = \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k)$$

$$\therefore\ f(a+h, b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(a,b)$$

$$+\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(a,b) + \dots + \frac{1}{(n-1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n-1} f(a,b) + R_n$$

where, $R_n = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(a+\theta h, b+\theta k), 0 < \theta < 1$

Here, R_n is called the remainder after n terms.

Part II We have, $f(x, y) = \tan^{-1}(xy)$

Let us expand $f(x, y)$ near the point $(1, -1)$ $f(0.9, -1.2) + f(1-0.1, -1-0.2)$

$$= f(1,-1) + \left[(-0.1)\frac{\partial f}{\partial x} + (-0.2)\frac{\partial f}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(-0.1)^2\frac{\partial^2 f}{\partial x^2} + 2(-0.1)(-0.2)\frac{\partial^2 f}{\partial x \partial f} + (-0.2)^2\frac{\partial^2 f}{\partial y^2}\right] + \dots \qquad \dots\text{(i)}$$

		$x = 1,\ y = -1$
$f(x, y)$	$\tan^{-1} xy$	$-\frac{\pi}{4}$
$\frac{\partial f}{\partial x}$	$\frac{y}{1+x^2y^2}$	$-\frac{1}{2}$
$\frac{\partial f}{\partial y}$	$\frac{x}{1+x^2y^2}$	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial x^2}$	$\frac{-2xy}{(1+x^2y^2)^2}$	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{1-x^2y^2}{(1+x^2y^2)}$	0
$\frac{\partial^2 f}{\partial y^2}$	$\frac{-x(2x^2y)}{(1+x^2y^2)^2}$	$\frac{1}{2}$

On putting these values in Eq. (i), we get

$$f(0.9, -1.2) = -\frac{\pi}{4} + (-0.1)\left(-\frac{1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)$$

$$+\frac{1}{2}\left[(-0.1)^2\left(\frac{1}{2}\right) + 2(-0.1)\,(-0.2)(0) + (-0.2)^2\left(\frac{1}{2}\right)\right] + \dots$$

$$= \frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2}(0.005 + 0.02)$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + 0.0125$$

$$= -0.823$$

Q 2. Write down Taylor's theorem for the function of two variables. Expand the function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ in the neighbourhood of the point (1,1) upto second degree terms using Taylor's theorem. Hence, compute $f(1.1, 0.9)$ approximately. **(2016, 13, 12, 06)**

Sol. **Part I Statement** See the solution of Q. 1 of Very Short Answer Questions.

Part II

		$x = 1,\ y = 1$
$f(x, y)$	$\tan^{-1}\left(\frac{y}{x}\right)$	$\frac{\pi}{4}$
$\frac{\partial f}{\partial x}$	$\frac{-y}{x^2 + y^2}$	$-\frac{1}{2}$
$\frac{\partial f}{\partial y}$	$\frac{x}{x^2 + y^2}$	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial x^2}$	$\frac{2xy}{(x^2 + y^2)^2}$	$\frac{1}{2}$
$\frac{\partial^2 f}{\partial y^2}$	$\frac{-2xy}{(x^2 + y^2)^2}$	$-\frac{1}{2}$
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{y^2 - x^2}{(x^2 + y^2)^2}$	0

By Taylor's theorem,

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

Here, $a = 1,\ b = 1$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2}$$
$$+ \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 2(x-1)(y-1)(0) + (y-1)^2\left(-\frac{1}{2}\right)\right] + \ldots$$

$$\tan^{-1}\left(\frac{y}{x}\right)=\frac{\pi}{4}-\frac{1}{2}(x-1)+\frac{1}{2}(y-1)+\frac{1}{4}(x-1)^2-\frac{1}{4}(y-1)^2 \quad \ldots(i)$$

Now, $x-1=1.1-1=0.1,\ y-1=0.9-1=-0.1$

$$\therefore f(1.1, 0.9)=\frac{\pi}{4}-\frac{1}{2}(0.1)-\frac{1}{2}(-0.1)+\frac{1}{4}(0.1)^2-\frac{1}{4}(-0.1)^2=0.8862-0.1=0.7862$$

Q 3. State and prove Taylor's theorem for the function of two variables. Expand the function $x^2y+3y-2$ in powers of $(x-1)$ and $(y+2)$. **(2015, 14, 13, 10)**

Sol. **Part I** See the Part I of Q. 1.

Part II See the solution of Q. 3 of Very Short Answer Questions.

Q 4. Expand $\sin x$ in powers of $x-\frac{\pi}{2}$. **(2017)**

Sol. Let $f(x)=\sin x$

Now, $\sin x = f(x)$

$$=f\left[\frac{\pi}{2}+\left(x-\frac{\pi}{2}\right)\right]$$

$$=f\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)}{1!}f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)+\ldots \quad \ldots(i)$$

[by Taylor's series]

Then,
$$f(x)=\sin x \Rightarrow f\left(\frac{\pi}{2}\right)=1$$
$$f'(x)=\cos x \Rightarrow f'\left(\frac{\pi}{2}\right)=0$$
$$f''(x)=-\sin x \Rightarrow f''\left(\frac{\pi}{2}\right)=-1$$
$$f'''(x)=-\cos x \Rightarrow f'''\left(\frac{\pi}{2}\right)=0$$
$$f^{iv}(x)=\sin x \Rightarrow f^{iv}\left(\frac{\pi}{2}\right)=1$$
$$\vdots \qquad \vdots$$

n putting these values in Eq. (i), we get

$$\sin x = 1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)}{4!}-\ldots$$