

1. **Simultaneous Equations** The general form of simultaneous equations of first order having three variables are

$$P_1 dx + Q_1 dy + R_1 dz = 0$$

and

$$P_2 dx + Q_2 dy + R_2 dz = 0$$

where the coefficients are functions of x, y and z . Solving these equations simultaneously, we get

$$\frac{dx}{Q_1 R_2 - Q_2 R_1} = \frac{dy}{R_1 P_2 - R_2 P_1} = \frac{dz}{P_1 Q_2 - P_2 Q_1}$$

which is of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where P, Q and R are functions of

x, y, z .

Q 1. Solve $\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$.

(2004)

Or Solve the following system of differential equation.

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x}$$

(2017)

Sol. The given differential equation is

$$\frac{dx}{z} = \frac{dy}{0} = \frac{dz}{-x} \quad \dots(i)$$

Taking first and last fractions, we have

$$\frac{dx}{z} = \frac{dz}{-x} \Rightarrow xdx + zdz = 0 \Rightarrow x^2 + z^2 = c_1 \quad \dots(ii)$$

Also, from second fraction, we have

$$dy = 0 \Rightarrow y = c_2 \quad \dots(iii)$$

Hence, the required general solution is given by Eqs. (ii) and (iii).

Q 2. Solve $\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$.

(2016)

Sol. The given differential equation is

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

Taking first and second fractions, we have

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow xdx = ydy$$

$$\Rightarrow x^2 - y^2 = c_1 \quad \dots(i)$$

Taking first and last fractions, we have

$$\frac{dx}{yz} = \frac{dz}{xy} \Rightarrow xdx = zdz \Rightarrow x^2 - z^2 = c_2 \quad \dots(ii)$$

From Eqs. (i) and (ii), the complete solution is

$$\phi(x^2 - y^2, x^2 - z^2) = 0$$

Q 4. Find the auxiliary equation for the following system of differential equations.

$$Dx + wy = 0, wx - Dy = 0$$

(2014)

Sol. The given differential equations are

$$Dx + wy = 0 \quad \dots(i)$$

and $wx - Dy = 0 \quad \dots(ii)$

From Eq. (i), we have $D^2x + wDy = 0$

$$\Rightarrow D^2x + w(wx) = 0 \quad \text{[from Eq. (ii)]}$$

$$\Rightarrow (D^2 + w^2)x = 0$$

Hence, the auxiliary equation is $m^2 + w^2 = 0$.

Q 2. Solve $\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$.

(2009, 1995)

Sol. The given differential equation is

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

Choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get

$$\frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \text{Each fraction}$$

$$\Rightarrow \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integrating, we get

$$\log x + \log y + \log z = \log c \Rightarrow xyz = c_1 \quad \dots(i)$$

Again, choosing x, y, z as multipliers, we get

$$\frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \text{Each fraction}$$

$$\Rightarrow xdx + ydy + zdz = 0$$

On integrating, we get

$$x^2 + y^2 + z^2 = c_2 \quad \dots(ii)$$

Hence, the complete solution is given by Eqs. (i) and (ii).

Q 3. Solve $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)}$.

(2013, 07)

Sol. The given differential equation is

$$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)}$$

Taking first and second fractions, we have

$$\frac{dx}{1} = \frac{dy}{-2} \Rightarrow dy + 2dx = 0$$

On integrating, we get

$$y + 2x = c_1 \quad \dots(i)$$

Taking first and last fractions, we get

$$dx = \frac{dz}{3x^2 \sin(y + 2x)}$$

$$\Rightarrow dz - 3x^2 \sin c_1 dx = 0$$

On integrating, we get

$$z - x^3 \sin c_1 = c_2$$

$$\Rightarrow z - x^3 \sin (y + 2x) = c_2$$

Hence, the complete solution is given by Eqs. (i) and (ii).

[from Eq. (i)]

...(ii)

Q 4. Solve $\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$.

(2017, 15, 12, 06)

Sol. The given differential equation is

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Then, $\frac{dx - dy}{(x - y)(x + y + z)} = \frac{dy - dz}{(y - z)(x + y + z)} = \frac{dz - dx}{(z - x)(x + y + z)}$

Taking first and second fractions, we have

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

On integrating, we get

$$\log (x - y) = \log (y - z) + \log c_1$$

$$\Rightarrow \frac{x - y}{y - z} = c_1 \quad \dots(i)$$

Taking second and third fractions, we have

$$\frac{dy - dz}{y - z} = \frac{dz - dx}{z - x}$$

On integrating, we get

$$\log (y - z) = \log (z - x) + \log c_2$$

$$\Rightarrow \frac{y - z}{z - x} = c_2 \quad \dots(ii)$$

Hence, the complete solution is given by Eqs. (i) and (ii).

Q 6. Solve $(y + z)dx + (z + x)dy + (x + y)dz = 0$. (2010)

Sol. The given differential equation is

$$(y + z)dx + (z + x)dy + (x + y)dz = 0 \quad \dots(i)$$

On comparing Eq. (i) with $Pdx + Qdy + Rdz = 0$, we get

$$P = y + z, Q = z + x \text{ and } R = x + y$$

$$\text{Now, } P \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y} \right) + Q \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right) + R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

$$= (y + z)(1 - 1) + (z + x)(1 - 1) + (x + y)(1 - 1) = 0$$

Thus, the condition of integrability is satisfied.

Hence, the given equation is integrable.

From Eq. (i), we have

$$(xdy + ydx) + (ydz + zdy) + (zdx + xdz) = 0$$

$$\Rightarrow d(xy) + d(yz) + d(zx) = 0$$

On integrating, we get

$$xy + yz + zx = c$$

Q 7. Solve the following simultaneous equations.

$$\frac{dx}{dt} = 7x - y, \quad \frac{dy}{dt} = 2x + 5y \quad (2015)$$

Sol. The given differential equations can be written as

$$(D - 7)x + y = 0 \quad \left[\text{where, } D \equiv \frac{d}{dt} \right] \dots(i)$$

$$-2x + (D - 5)y = 0 \quad \dots(ii)$$

and

On eliminating y from Eqs. (i) and (ii), we get

$$[(D - 5)(D - 7) + 2]x = 0 \Rightarrow (D^2 - 12D + 37)x = 0$$

Its auxiliary equation is

$$m^2 - 12m + 37 = 0 \Rightarrow m = 6 \pm i$$

$$x = e^{6t} (c_1 \cos t + c_2 \sin t) \quad \dots(iii)$$

On differentiating Eq. (iii) w.r.t. t , we get

$$\frac{dx}{dt} = 6e^{6t} (c_1 \cos t + c_2 \sin t) + e^{6t} (-c_1 \sin t + c_2 \cos t)$$

Now, from Eq. (i), we have

$$y = 7x - Dx = e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t) \\ = e^{6t}[(c_1 + c_2) \sin t + (c_1 - c_2) \cos t] \quad \dots(\text{iv})$$

Hence, the required solution is given by Eqs. (iii) and (iv).

Long Answer Questions

Q 1. Solve $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$.

(2011, 08)

Sol. The given differential equation is

$$\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$$

We have,

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} = \frac{dz}{z}$$

(I) (II) (III)

Taking first and third fractions, we have

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx+dy}{\sqrt{2} \sin\left(x+y+\frac{\pi}{4}\right)} \\ = \frac{1}{\sqrt{2}} \operatorname{cosec}\left(x+y+\frac{\pi}{4}\right) (dx+dy)$$

On integrating, we get

$$\sqrt{2} \log z = \log \tan\left(\frac{x+y+\pi/4}{2}\right) + \log c_1$$

$$\Rightarrow z^{\sqrt{2}} = c_1 \tan\left(\frac{x+y}{2} + \frac{\pi}{8}\right) \quad \dots(\text{i})$$

Now, taking first and second fractions, we get

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \\ \frac{\{-\sin(x+y)+\cos(x+y)\}}{\cos(x+y)+\sin(x+y)} (dx+dy) = dx-dy$$

On integrating, we get

$$\log\{\sin(x+y)+\cos(x+y)\} = x-y + \log c_2$$

$$\Rightarrow \sin(x+y)+\cos(x+y) = c_2 e^{x-y}$$

$$\Rightarrow e^{y-x}\{\sin(x+y)+\cos(x+y)\} = c_2 \quad \dots(\text{ii})$$

Hence, the complete solution is given by Eqs. (i) and (ii).