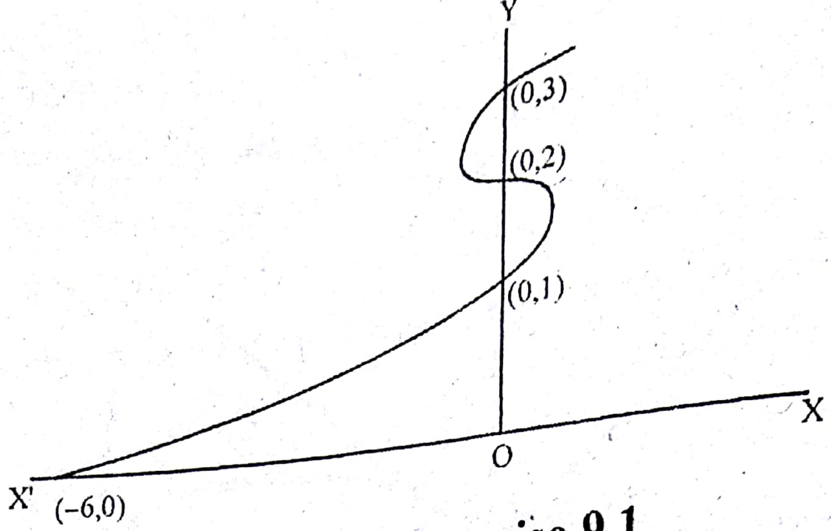


Excercise 9.1

1. Trace the curve $y^2(2a - x) = x^3$ [GKP, 2012]
2. Trace the curve $x^5 + y^5 = a^2xy$
3. Trace the curve $y^2(x^2 + y^2) - 4x(x^2 + 2y^2) + 16x^2 = 0$
4. Trace the curve $y^2(a + x) = x^2(a - x)$ [GKP, 2002]
5. Trace the curve $x^4 - y^4 = x^2y + x^2 - y^2$
6. Trace the curve $y^2 = (x - 2)^2(x - 5)$
7. Trace the curve $y^2(x + 3a) = x(x - a)(x - 2a)$, $a > 0$
8. Trace the curve $y^2(x - a) = x^2(x + a)$

Tracing the polar curves.

We shall adopt the following procedure for tracing the curve :

1. Symmetry. (i) If the equation of curve remains unchanged when θ is changed into $-\theta$ then the curve is symmetrical about the initial line.

(ii) If the equation of curve remains unchanged when θ is changed into $(\pi - \theta)$ then the curve has symmetry about the line $\theta = \frac{\pi}{2}$.

(iii) If the equation of the curve remains unchanged when θ is changed into $\frac{\pi}{2} - \theta$ then the curve has symmetry about the line $\theta = \frac{\pi}{4}$.

(iv) If the equation of curve remains unchanged when r changed into $-r$ then the curve is symmetrical about the pole.

2. Tangent at the pole. Find if the pole lies on the curve. For this put $r = 0$ in the equation of the curve and find the value of θ. If the value of θ are real then the curve passes through pole. If $r = 0$ for $\theta = \alpha$, then the line $\theta = \alpha$, is the tangent to the curve at pole. If pole is a singular point, find its nature.

3. Asymptotes. Find the asymptote of the curve, if any.

4. Regions of the curve. Find the values of θ for which r becomes imaginary. If r is imaginary for $\alpha < \theta < \beta$, then the curve does not lie in the region bounded by the lines $\theta = \alpha$ and $\theta = \beta$.

5. Some points of the curve. Prepare a table for corresponding value of r and θ for the curve.

6. Direction of tangent. Find φ using the formula $\tan\phi = r\frac{d\theta}{dr}$. This gives direction of tangent with radius vector for some special values of θ, say $\theta = 0, \frac{\pi}{2}, \pi$.

Note : If n is odd, the curve $r = a\cos n\theta$ (or $r = a\sin n\theta$) has n loops; but if n is even the curve has 2n loops.

Example 7. Trace the curve $r = a + b\cos\theta$ when (i) $a > b$, (ii) $a < b$, (iii) $a = b$.

Solution. Case (I). a > b.

(i) Equation of curve remains unchanged when θ is changed into −θ, so the curve has symmetry about initial line.

(ii) Putting $r = 0$ in the equation of curve we get $0 = a + b\cos\theta$, i.e. $\cos\theta = -\frac{a}{b}$. Since $a > b$ so the above relation does not give any real value of θ for $r = 0$. Hence the curve does not pass through the pole.

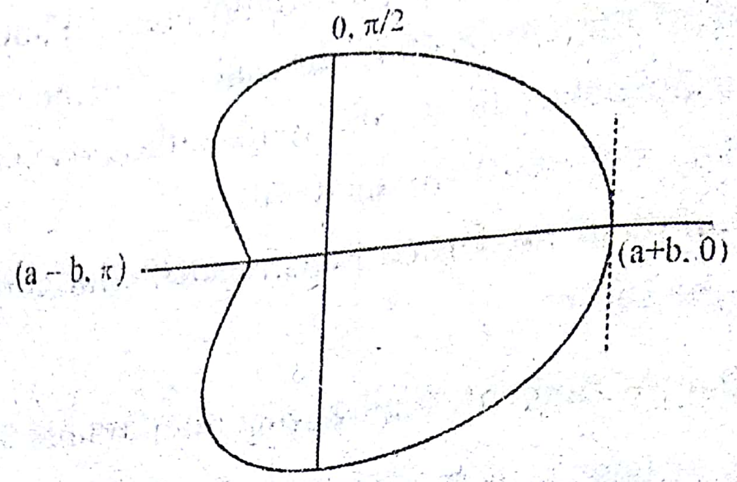
(iii) For different values of θ, the corresponding values of r are given in the following table :

θ	0	π/3	π/2	2π/3	π
r	a + b	a + b/2	2	a − b/2	a - b

(iv) The equation of curve gives $\frac{dr}{d\theta} = -b\sin\theta$.

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = \frac{a + b\cos\theta}{-b\sin\theta}$$

When $\theta = 0$ $\tan\phi = \infty$, i.e. $\phi = \pi/2$. Hence at point $(a + b, 0)$ tangent is perpendicular to radius vector. Considering symmetry about initial line and plotting the above points and direction of tangent the shape of the curve is as shown in the figure.



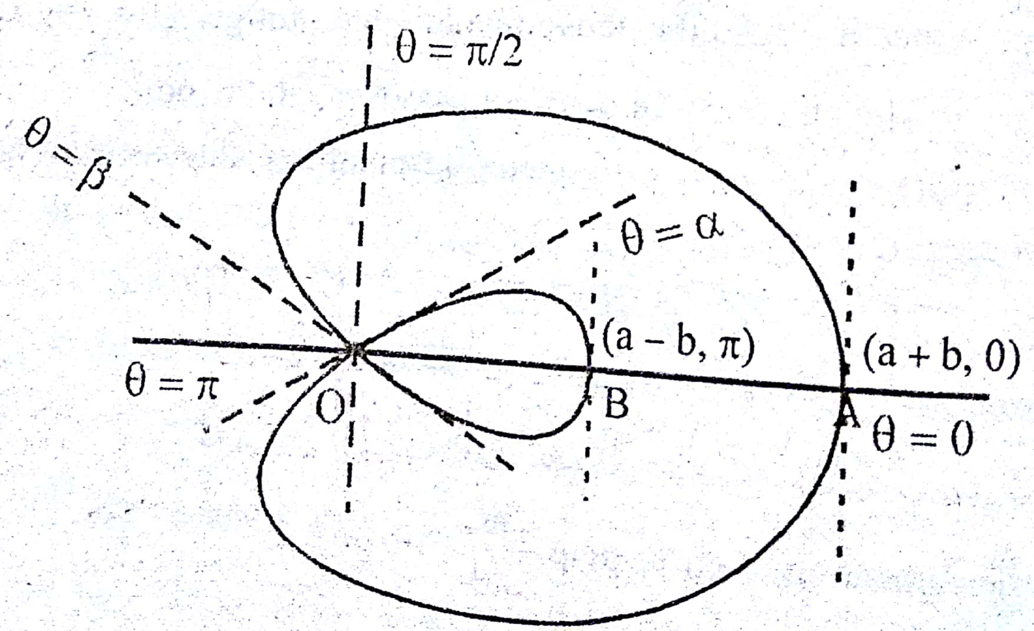
Case II. a < b.

(i) The curve is symmetrical about the initial line.

(ii) For $r = 0$, $0 = a + b\cos\theta$ or $\theta = \cos^{-1}(-a/b)$. Since $a < b$, therefore we get two real values of θ between $\theta = 0$ and $\theta = 2\pi$, say $\theta = \alpha$, $\theta = \beta$. Thus tangents at the pole are $\theta = \alpha$ and $\theta = \beta$.

(iii) For different values of θ, the values of r are given in following table

θ	0	$\pi/3$	$\pi/2$	$2\pi/3$	π
r	a + b	a + b/2	a	a − b/2	a - b



(iv) $\tan\phi = \dfrac{a + b\cos\theta}{-b\sin\theta}$,

thus when θ = 0 ...

Tracing of curve with parametric equations.

Method I : Find the Cartesian Equation of the curve by eliminating the parameter between the parametric equations of the curve and trace it.

Method II : Find $\frac{dy}{dx}$ by $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Then find ψ from $\tan\psi = \frac{dy}{dx}$ at different points and prepare table in x, y, t and ψ and trace the curve.

Example 9. Trace the curve

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}, \quad y = a\sin t$$

Solution :

(i) We have $\frac{dx}{dt} = -a\sin t + \frac{a}{2} \cdot \frac{1}{\tan^2\frac{t}{2}} \cdot 2\tan\frac{t}{2} \cdot \sec^2\frac{t}{2} \cdot \frac{1}{2}$

$$= -a\sin t + \frac{a}{2\tan\frac{t}{2}\cos^2\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= a\left(\frac{1-\sin^2 t}{\sin t}\right)$$

$$= \frac{a\cos^2 t}{\sin t},$$

and $\frac{dy}{dt} = a\cos t$

$$\therefore \quad \tan\psi = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \tan t$$

i.e. $\psi = t.$

The table between t, x, y and ψ is as follows :

t	$-\pi$	$-\pi/2$	0	$\pi/2$	π
x	∞	0	$-\infty$	0	∞
y	0	$-a$	0	a	0
ψ	$-\pi$	$-\pi/2$	0	$\pi/2$	π

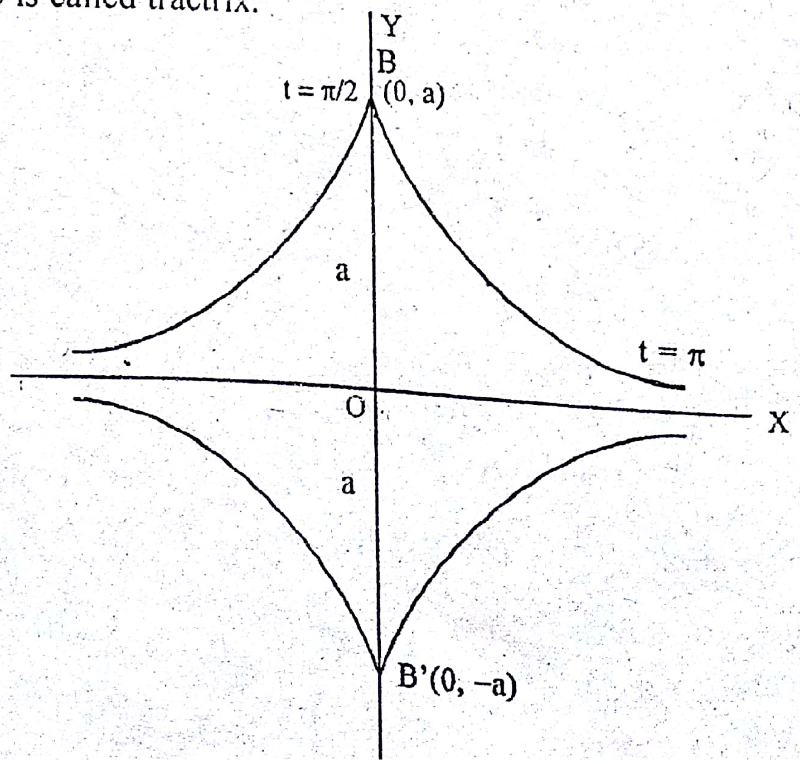
(ii) For t if we put $-t$, we get $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = -a\sin t$.

This shows that the curve is symmetrical about x-asix. If we put $\pi - t$ in place of t, we get $x = -a\cos t - \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = a\sin t$. So, for every value of y we get two equal and opposite values of x. Hence the curve is symmstrical about y-axis.

(iii) When $t = 0$, $x \to -\infty$ and $y \to -\infty$, showing that $y = 0$ i.e., x-axis is an asymptote of the curve.

(iv) When $t = \pi/2$, $x = 0$, $y = a$ and $\frac{dy}{dx} = \infty$. This shows that the curve passes through $(0, a)$ and tangent at this point is y-axis.

Considering all the points the shape of the curve is as shown in figure. This curve is called tractrix.



Areas of Curves (Quadrature)

1. Areas of curves given by Cartesian equations.

Theorem : If f(x) is a single-vaued and continuous function of x in the interval (a, b), then the area bounded by the curve y = f(x), the x-axis, and the ordinates at x = a and x = b is

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_a^b y\,dx.$$

Proof : Let AD be the curve given by the equation y = f(x), and BA and CD be the ordinates at x = a and x = b. We are now to find area ABCD which is bounded by y = f(x), x = a, x = b and x-axis.

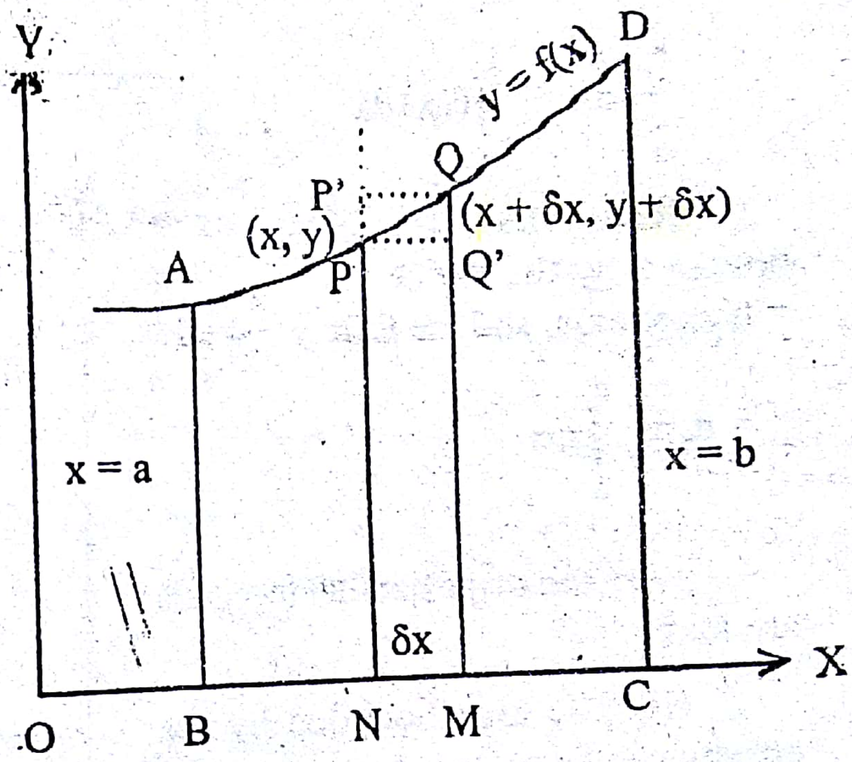


Fig. 6.1

Let P(x, y) be any point on the curve and Q(x + δx, y + δy) be neighbouring point of the curve. Draw ordinates PN and QM, then PN = y, QM = y + δy and NM = δx.

Let A denote the area BNPA, then area ABMQ will be A + δA,

i.e. area NMQP = δA.

Now area NMQ'P = y δx

and area NMQP' = (y + δy) δx.

Since area NMQP lies between the areas NMQ'P and NMQP'

∴ $\frac{\delta A}{\delta x}$ lies between y and y + δy.

In the limit when Q→P, i.e. δx→0 and δy→0, we have

$$\frac{dA}{dx} = y,$$

or $dA = y\,dx.$

Integrating between the limits a to b, we have

$\int_a^b y\,dx = [A]_a^b$ = (area when x = b) - (area when x = a)

= Area ABCD – 0.

Hence the required area between the curve y = f(x), the axis of x, and ordinates at x=a and x=b, is

$$\int_a^b y\,dx = \int_a^b f(x)\,dx.$$

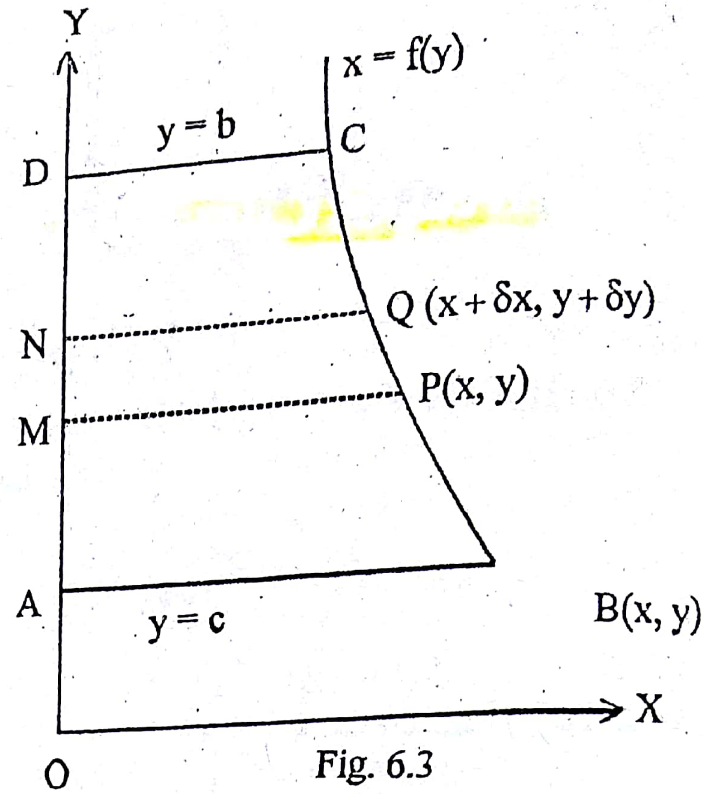


Fig. 6.3

Remarks. (i) The area bounded by the curve x = f(y), y-axis, and the lines y = c and y = d, is $\int_c^d x\,dy$.

Here the strips parallel to x-axis are taken.

(ii) The area bounded by the curves $y = f_1(x)$, $y = f_2(x)$, the lines x = a and x = b, is

$$\int_a^b [f_2(x) - f_1(x)]\,dx.$$

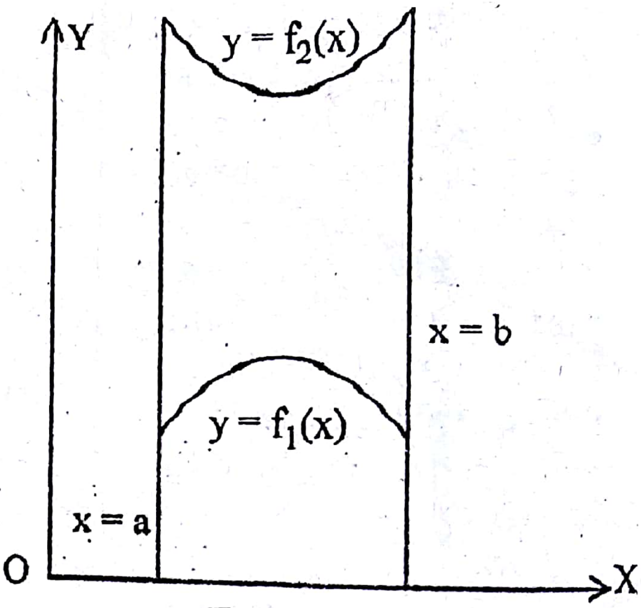
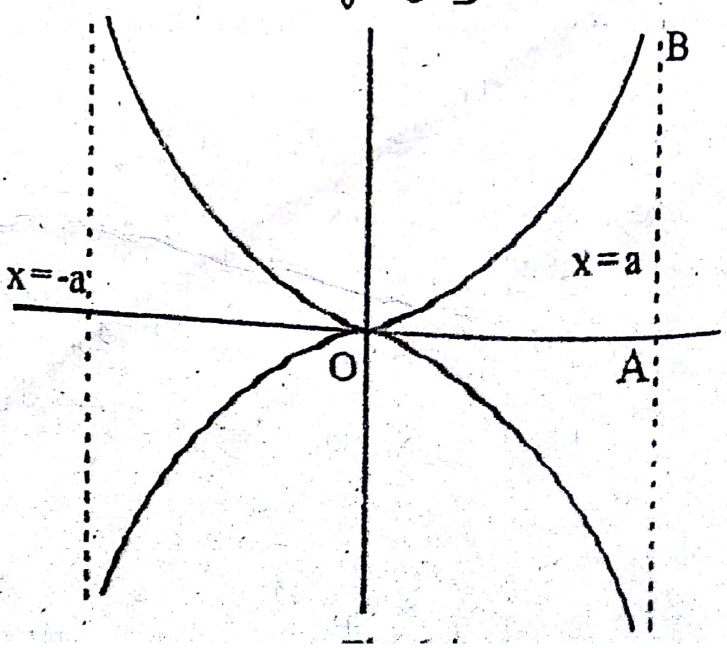


Fig. 6.3

(iii) If the curve is symmetrical about x-axis, then find the area above the axis of x and multiply it by 2.



Example 1 : Find the whole area between the curve

$$= \left[\frac{y^2}{8} + \frac{y}{4} - \frac{y^3}{6}\right]_{-1/2}^{1}$$

$$= \frac{5}{24} + \frac{7}{96} = \frac{9}{32} \text{ square units.}$$

Example 4. Prove that the whole area between the four infinite branches of the tractrix

$x = a\cos t + \frac{1}{2} a \log \tan^2 \frac{t}{2}$,

$y = a \sin t$ is πa^2

[GKP, 1997]

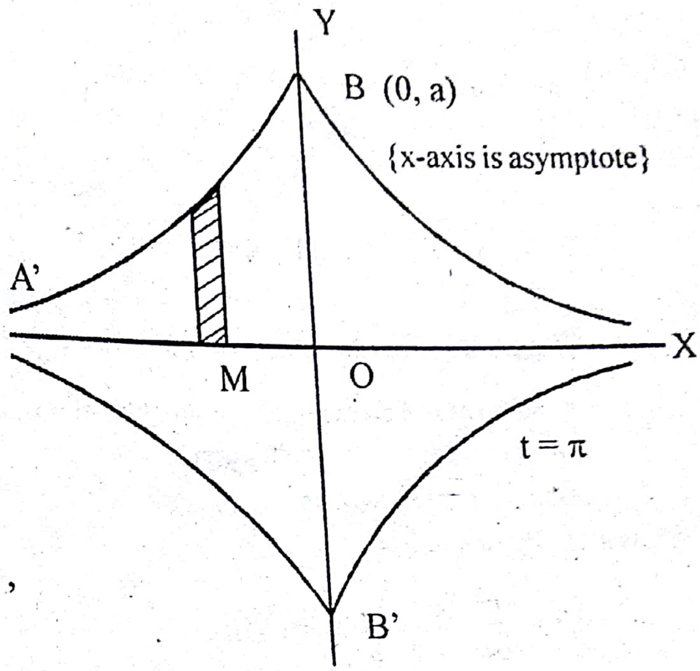


Fig. 6.7

Solution.

The given curve is

$x = a\cos t + \frac{1}{2} a \log \tan^2 \frac{t}{2}$,

$y = a \sin t$.

The required area

$= 4 \times$ Area OBA'

$$= 4 \int_{t=0}^{\pi/2} y \frac{dx}{dt} dt \qquad \text{.....(i)}$$

Now $$\frac{dx}{dt} = -a\sin t + \frac{1}{2} a \frac{2\tan\frac{t}{2}\sec^2\frac{t}{2}}{\tan^2\left(\frac{1}{2}t\right)} \frac{1}{2}$$

$$= -a \sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}} = -a\sin t + \frac{a}{\sin t}$$

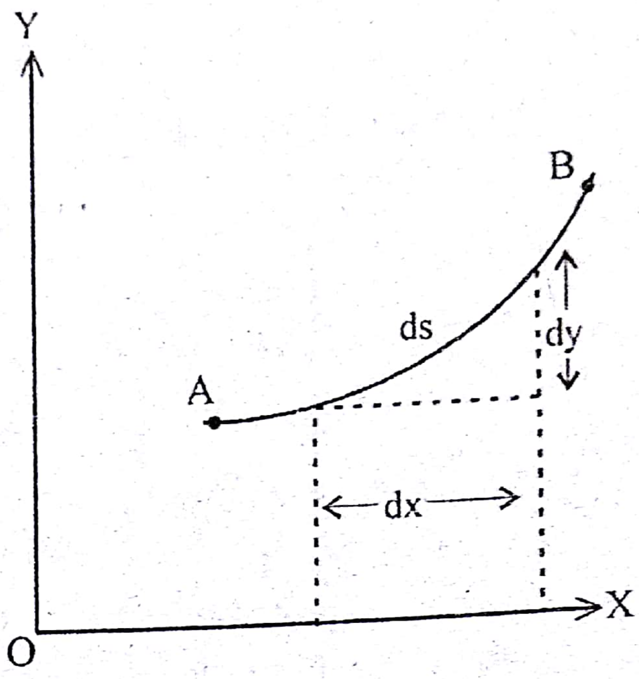
$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

Hence from (i),

$$\text{Required area} = 4 \int_0^{\pi/2} a \sin t \frac{a\cos^2 t}{\sin t} dt$$

Lengths of curves (Rectification)

1. **Length of arcs of curves.** If s be the length of an arc of the curve y = f(x) measured from a fixed point A(a, b) to a point B(c, d) on it, then



(i) $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, so

so that $s = \int_a^c \sqrt{\left(1 + \frac{dy}{dx}\right)^2}\, dx$;

(ii) $\frac{ds}{dy} = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}$,

so that $s = \int_b^d \sqrt{\left(1 + \frac{dx}{dy}\right)^2}\, dy$.

If the equation of curve is in polar from r = f (θ) then the length of the curve r = f(θ) from point A(r_1, α) to B(r_2, β) is given by

[**Hint :** In Cartesian to polar transformation dx → dr, dy → rdθ]

(iii) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$, so that $s = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$;

(iv) $\frac{ds}{dr} = \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}$, so that $s = \int_{r_1}^{r_2} \sqrt{1 + \left(\frac{r\,d\theta}{dr}\right)^2}\, dr$.

If the equation of curve is in parametric form $x = f_1(t)$, $y = f_2(t)$, then

(v) $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$, so that $s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

$$= 4a \int_0^{\pi} \cos\frac{t}{2}\sqrt{\left(\cos^2\frac{t}{2} + \sin^2\frac{t}{2}\right)}\,dt$$

$$= 4a \int_0^{\pi} \cos\frac{t}{2}\,dt$$

$$= 4a \cdot 2\left[\sin\frac{t}{2}\right]_0^{\pi}$$

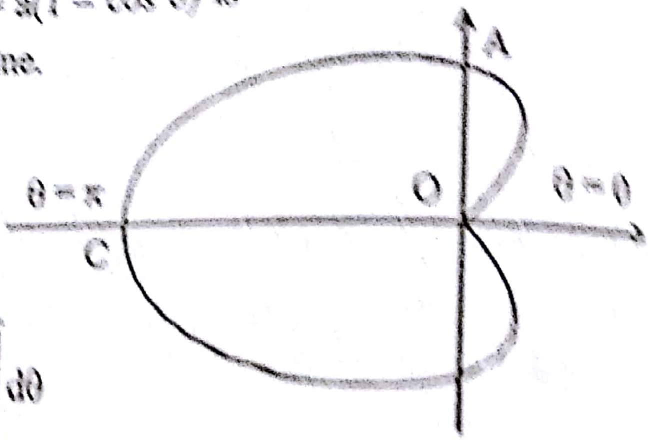
$$= 8a.$$

Example 3. Find the perimeter of the cardioid $r = a(1 - \cos\theta)$.

[GKP, 1997]

Solution. The cardioid $r = a(1 - \cos\theta)$ is symmetrical about initial line.

We have $\dfrac{dr}{d\theta} = a\sin\theta.$



Length of cardioid

$= 2$ arc OAC

$$= 2\int_0^{\pi}\sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}\,d\theta$$

$$= 2\int_0^{\pi}\sqrt{\left[a^2(1-\cos\theta)^2 + a^2\sin^2\theta\right]}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{(1 + \cos^2\theta - 2\cos\theta + \sin^2\theta)}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{2(1-\cos\theta)}\,d\theta$$

$$= 2a\int_0^{\pi}\sqrt{2 \cdot 2\sin^2\frac{\theta}{2}}\,d\theta$$

$$= 4a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta$$

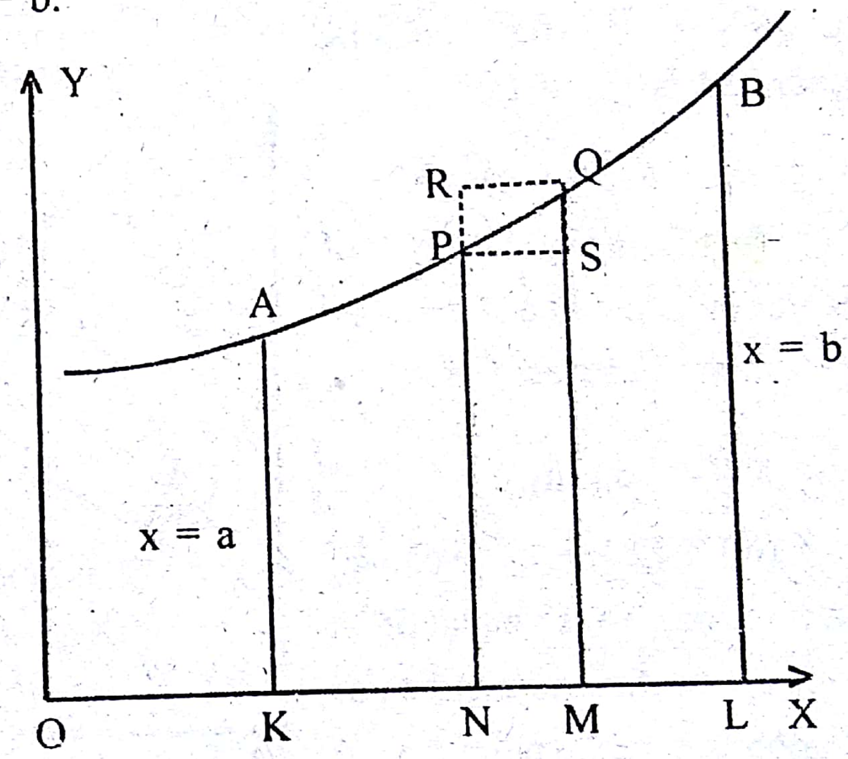
$$= 8a\left[-\cos\frac{\theta}{2}\right]_0^{\pi}$$

Volumes and Surfaces of Solids of Revolution

8.1. Volumes of Solids of Revolution.

Theorem. The volume of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a$, $x = b$, is $\int_a^b \pi y^2 \, dx$.

Proof. Let AB be the curve given by $y = f(x)$ and AK and BL be two given ordinates $x = a$ and $x = b$.



Let (x, y) and Q $(x + \delta x, y + \delta y)$ be two neighbouring points of arc AB. From P and Q draw PN and QM perpendiculars on x-axis. From P draw PS and from Q draw QR perpendiculars to QM and PN produced respectively.

Let the volumes of the solid generated by the revolution of area AKNPA and AKMQPA about the x-axis be V and $V + \delta V$ respectively. Then δV is the volume of the solid generated by the revolution of area PNMQP about the x-axis.

Now $PN = y$, $QM = y + \delta y$ and $NM = \delta x$.

$\therefore$ Volume generated by revolving the area PNMS

$= \pi y^2 \delta x$, and volume generated by revolving the area RNMQ

$= p(y + \delta y)^2 \delta x$.

Now δV, i.e. volume generated by revolving the area PNMQ, lies between the volumes generated by the area RNMQ and PNMS, i.e.,

δV lies between $\pi(y + \delta y)^2 \delta x$ and $\pi y^2 \delta x$

or $\dfrac{\delta V}{\delta x}$ lies between $\pi(y + \delta y)^2$ and πy^2.

In the limiting case when $Q \to P$, i.e. $\delta x \to 0$, we get

$$\frac{dV}{dx} = \pi y^2.$$

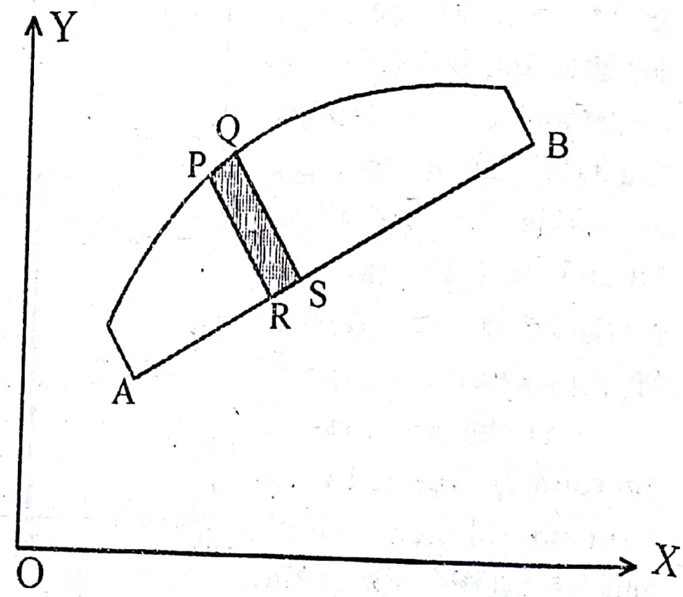
Integrating between $x = a$ and $x = b$, we get required volume

$$V = \pi \int_a^b y^2 \, dx.$$

Remarks. (i) If the area bounded by the curve $x = f(y)$, the lines $y = c$, $y = d$ and y-axis is revolved about y-axis, then volume of the solid thus generated is given by

$$V = \pi \int_c^d x^2 \, dy.$$

(ii) If the area bounded by $y = c$, $x = a$, $x = b$ and $y = f(x)$ is revolved about $y = c$, then volume generated is given by



$$\pi \int_a^b (y - c)^2 \, dx$$

(iii) Similarly if the axis of revolution is the line $x = a$, then the volume generated is

$$\pi \int_c^d (x - a)^2 \, dy.$$

(iv) If the axis of revolution is a line AB, whose equation is $y = mx + c$, then perpendicular PR upon it from the point $P(x, y)$ of the curve is

$$\frac{y - mx - c}{\sqrt{1 + m^2}}.$$

The volume thus generated

$$= \pi \int (PR)^2 RS$$

$$= \pi \int (PR)^2 d(AR)$$

$$= \pi \int \left(\frac{y - mx - c}{\sqrt{1 + m^2}} \right)^2 \sqrt{(dx)^2 + (dy)^2}.$$

where the limit of integration extends from A to B.

... about the initial line, then the volume generated

$$= \pi \int_{\alpha}^{\beta} y^2\,dx$$

$$= \pi \int_{\alpha}^{\beta} y^2 \frac{dx}{d\theta}\,d\theta$$

$$= \pi \int_{\alpha}^{\beta} (r\sin\theta)^2 \frac{d}{d\theta}(r\cos\theta)\,d\theta,$$

where α, β are the values of θ corresponding to $x = a$ and $x = b$.

(vi) The volume V generated by revolving parametric curve $x = f_1(t)$, $y = f_2(t)$ about x-axis is given by

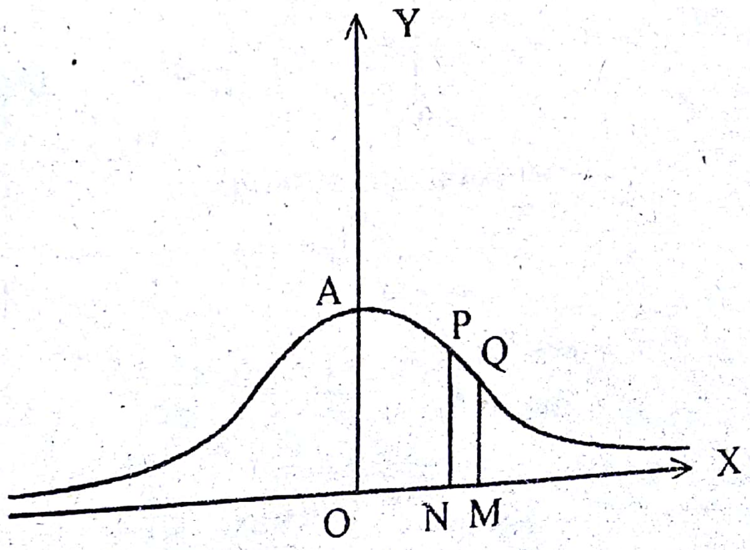
$$V = \pi \int_{t_1}^{t_2} y^2 \frac{dx}{dt}\,dt$$

$$= \pi \int_{t_1}^{t_2} \{f_2(t)\}^2 \frac{df_1(t)}{dt}\,dt,$$

where t_1 and t_2 correspond to $x = a$ and $x = b$.

Example 1. Find the volume of the solid generated by the revolution of curve $y = \dfrac{a^3}{(a^2+x^2)}$ about its asymptote. [GKP, 1999]

Solution : The equation of the curve is $y = \dfrac{a^3}{(a^2+x^2)}$, or, $x^2y = a^2(a - y)$.



The equation of asymptote is $y = 0$ i.e. x-axis and the curve is symmetrical about y-axis. So the volume V generated by revolving the curve about x-axis

$$= 2\pi \int_0^{\infty} y^2\,dx$$

$$= 2\pi \int_0^\infty \frac{a^6}{(a^2+x^2)^2} dx .$$

Putting $x = a \tan\theta$, $dx = a \sec^2\theta \, d\theta$, we get

$$V = 2a^6\pi \int_0^{\pi/2} \frac{a \sec^2\theta \, d\theta}{a^4(1+\tan^2\theta)^2}$$

$$= 2a^3\pi \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta} d\theta = 2a^3\pi \int_0^{\pi/2} \cos^2\theta \, d\theta$$

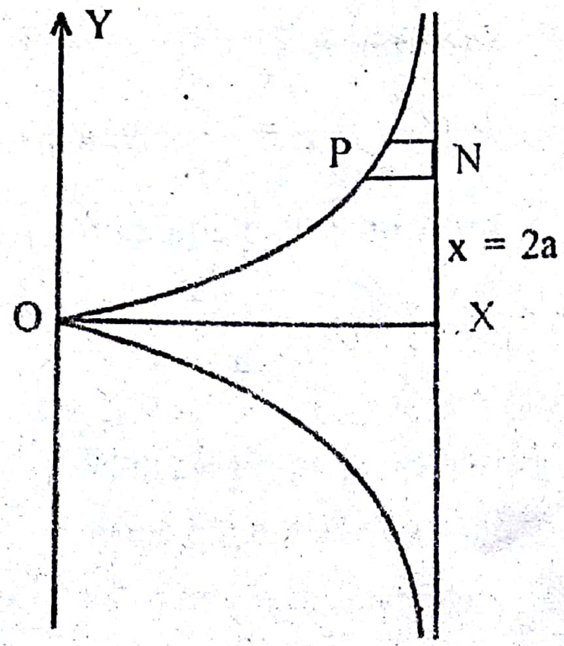
$$= 2\pi a^3 \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(2)} = \pi a^3 \frac{1}{2}\sqrt{\pi}\sqrt{\pi} = \frac{1}{2}\pi^2 a^3.$$

Example 2. Find the volume of the solid obtained by the revolution of cissoid $y^2(2a - x) = x^3$ about its asymptote. [GKP, 1996]

Solution. The curve is symmetrical about x-axis. The curve passes through the origin and $y^2 = 0$ is tangent at origin, i.e. and origin is a cusp. The asymptote is $x = 2a$.

If $P(x, y)$ be any point on the curve and PN is perpendicular from P on the asymptote, then

$$PN = 2a - x.$$



Hence the required volume

$$= \pi \int_{-\infty}^{\infty} (2a - x)^2 \, dy$$

$$= 2\pi \int_0^{2a} (2a - x)^2 \frac{dy}{dx} dx .$$

The equation of curve is

$$y^2 = \frac{x^3}{(2a - x)} .$$

$$\therefore \quad 2y \frac{dy}{dx} = \frac{(2a - x)3x^2 + x^3}{(2a - x)^2}$$

or $$2\frac{x^{3/2}}{\sqrt{(2a - x)}} \frac{dy}{dx} = \frac{6ax^2 - 2x^3}{(2a - x)^2} = \frac{2x^2(3a - x)}{(2a - x)^2}$$

or $\frac{dx}{...} \quad (2a - x)^{3/2}$

$$\therefore \quad \text{Volume} = 2\pi \int_0^{2a} \frac{(2a-x)^2\sqrt{x}\,(3a-x)}{(2a-x)^{3/2}}\,dx$$

$$= 2\pi \int_0^{2a} (3a - x)\sqrt{(2a-x)}\,\sqrt{x}\,dx$$

$$= 2\pi \int_0^{\pi/2} (3a - 2a\sin^2\theta)\sqrt{2a}\cos\theta\sqrt{2a}\sin\theta\; 4a\sin\theta\cos\theta\,d\theta,$$

putting $x = 2a\sin^2\theta$, $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 16a^3\pi \int_0^{\pi/2} (3\sin^2\theta\cos^2\theta - 2\sin^4\theta\cos^2\theta)\,d\theta = 2a^3\pi^2.$$

(using Walli's formula)

Example 3. Find the volume of the solid generated by revolution of the tractrix $x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2}$, $y = a\sin t$ about its asymptote.

Solution. The equations of the curve are

$$x = a\cos t + \frac{a}{2}\log\tan^2\frac{t}{2},\; y = a\sin t \qquad \text{.....(i)}$$

The curve is symmetrical about both the axes and $y = 0$ is asymptote of the curve. The shape of the curve is as shown in the figure.

From (i). we have

$$\frac{dx}{dt} = -a\sin t + \frac{a}{2}\frac{1}{\tan^2\frac{t}{2}}\, 2\tan\frac{t}{2}.\sec^2\frac{t}{2}.\frac{1}{2}$$

$$= -a\sin t + \frac{a}{2\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a\cos^2 t}{\sin t}.$$

The required volume

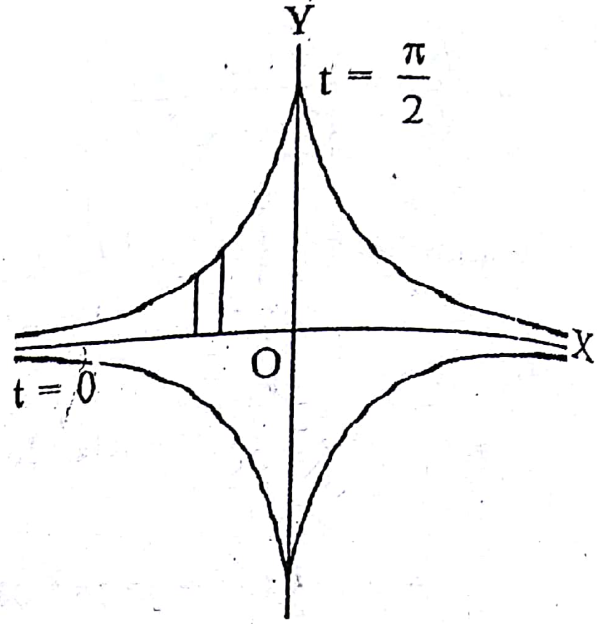
$$= 2\pi \int_0^{\frac{\pi}{2}} y^2 \frac{dx}{dt} dt$$

$$= 2\pi \int_0^{\frac{\pi}{2}} a^2 \sin^2 t \frac{a\cos^2 t}{\sin t} dt$$

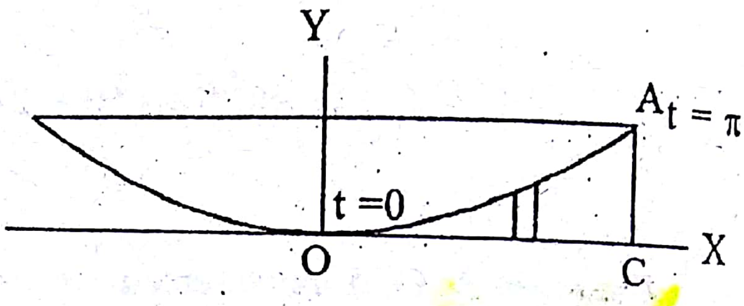
$$= 2\pi a^3 \int_0^{\frac{\pi}{2}} \sin t \cdot \cos^2 t \, dt$$

$$= \frac{2\pi a^3}{3}.$$

(using Walli's formula)



Example 4. Find the volume of the reel generated by the revolution of cycloid $x=a(t+\sin t), y=a(1-\cos t)$ about the tangent at vertex.



Solution. Here vertex is origin and tangent at the vertex is x-axis. Also from 0 to A, t varies from 0 to π.

∴ The required volume

= 2 × volume generated by revolution of the area OACO about x-axis

$$= 2\pi \int_0^{\pi} y^2 \frac{dx}{dt} dt$$

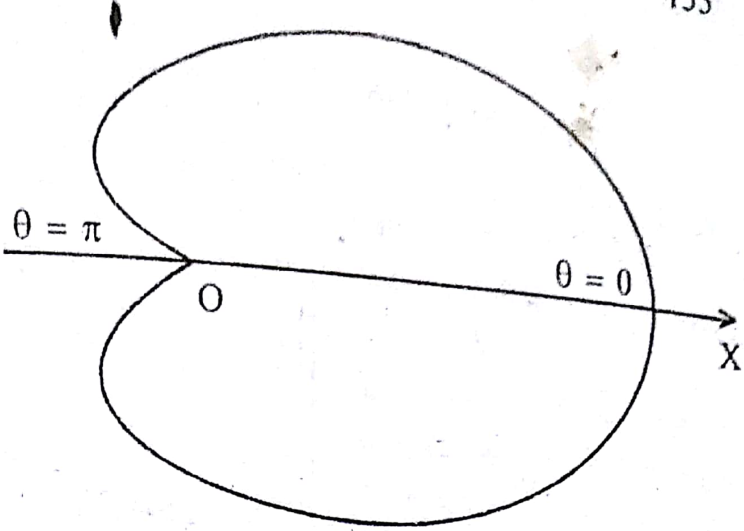
$$= 2\pi \int_0^{\pi} a^2 (1 - \cos t)^2 \, a(1+\cos t) \, dt$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2 \frac{t}{2}\right)^2 2\cos^2 \frac{t}{2} dt$$

$$= 32\pi a^3 \int_0^{\frac{\pi}{2}} \sin^4 \theta \cos^2 \theta \, d\theta$$

[putting $\frac{t}{2} = \theta$]

Example 7. Find the volume of the solid generated by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the initial line.



Solution. The required volume

$$= \frac{2\pi}{3}\int_0^{\pi} r^3 \sin\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3(1 + \cos\theta)^3 \sin\theta \, d\theta$$

$$= -\frac{2\pi}{3}a^3\int_2^0 t^3\, dt \qquad [\text{putting } 1 + \cos\theta = t, -\sin\theta\, d\theta = dt]$$

$$= \frac{2\pi a^3}{3}\left[\frac{t^4}{4}\right]_0^2$$

$$= \frac{\pi a^3}{6} \cdot 2^4 = \frac{8\pi a^3}{3}.$$

Exercise 8.2

1. Show that the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\frac{\pi^2 a^3}{4\sqrt{2}}$.

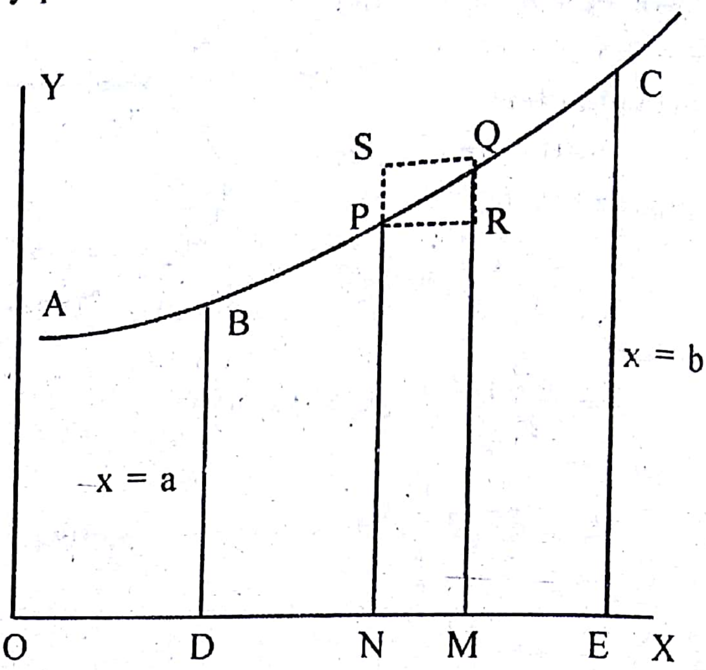
2. Show that the volume of the solid generated by revolving the lemniscate $r^2 = a^2 \cos 2\theta$ about a tangent at the pole is $\frac{\pi^2 a^3}{4}$.

3. The arc of cardoid $r = a(1 + \cos\theta)$ specified by $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ is revolved about the line $\theta = 0$. Find the volume thus generated.

4. Find the volume of anchor-ring generated by revolution of a circle of radius a about an axis in its own plane distant b from the centre when $b > a$.

8.4 Surfaces of solids of revolution.

Theorem. The surface of the solid generated by revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, the ordinates at $x = a$, x = b, and x-axis, is equal to $2\pi \int_{x=a}^{b} y\, ds$, where s is the length of arc measured from some fixed point A to any point (x, y).



Proof : Let BC be the position of the arc of the curve y = f(x) included between the ordinates x = a and x = b. Here P(x, y) and Q(x + δx, y + δy) are any two neighbouring points on the arc BC. From P and Q draw PN and QM perpendiculars on x-axis. PR and QS are perpendiculars from P on QM and from Q on NP produced respectively.

Let arc AP = s and arc PQ = δs. Let the area of surface of the solid generated by the revolution of arc BP and BQ about x-axis be denoted by A and A + δA respectively. Then δA is the area of surface generated by revolution of the arc PQ about x-axis.

Also the lines PR and SQ generate cylinders when they revolve about x-axis and the area of curved surface of these cylinders are 2πyδs and 2π(y + δy)δs respectively, because P and Q being two neighbouring points, we may take PR, SQ and arc PQ equal.

Also the surface generated by revolutions of arc PQ lies between these two surfaces, i.e. δA lies between 2πyδs and 2π(y + δy)δs, i.e. δA/δs lies between 2πy and 2π(y + δy).

In the limit when Q→P, we have

$$\frac{dA}{ds} = 2\pi y \quad \text{or} \quad dA = 2\pi y ds.$$

Integrating from x = a to x = b, the required surface is

$$2\pi \int_{x=a}^{x=b} y\, ds$$

Remark. (i) The surface of the solid generated by revolving about y-axis the area bounded by the curve x = f(y), the y-axis, and the lines y = c, y = d is equal to

$$2\pi \int_{y=c}^{d} x\, ds.$$

(ii) If the equation of curve is in the parametric forum $x = f_1(t)$, $y = f_2(t)$, and the curve revolves about axis of x, then surface of the solid of revolution is equal to

$$2\pi \int_{t=\alpha}^{\beta} y\, ds = 2\pi \int_{\alpha}^{\beta} f_2(t) \frac{ds}{dt}\, dt$$

$$= 2\pi \int_{\alpha}^{\beta} f_2(t) \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt,$$

where α and β are the values of t corresponding to the values $x = a$ and $x = b$ respectively.

(iii) If the equation of the curve is given in polar form, then the area of the surface of revolution about the axis of x is

$$2\pi \int y\, ds = 2\pi \int_{\theta=\alpha}^{\beta} r \sin\theta \frac{ds}{d\theta}\, d\theta$$

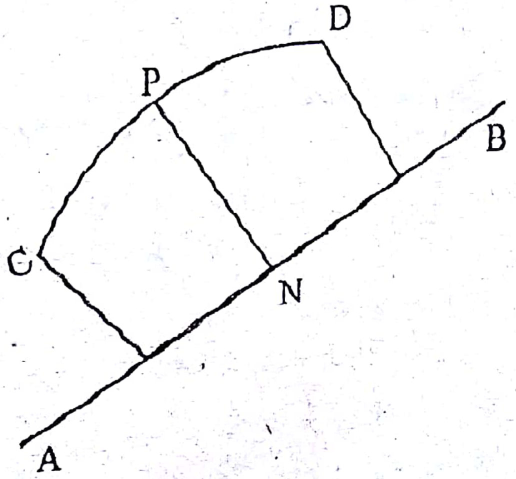
$$= 2\pi \int_{\alpha}^{\beta} (r \sin\theta) \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta,$$

where α and β are vectorial angles corresponding to $x = a$ and $x = b$ respectively.

(iv) If the curve revolves about any line, which is not one of the axes, then the surface of the solid genrated is

$$2\pi \int PN\, ds$$

where P is any point on the curve, A is any fixed point on the line AB, and PN is the perpendicular drawn from P to AB and s is the length of the arc CP.



Example 8. Find the surface of solid formed by revolution about x-axis of the loop of the curve $x = t^2$, $y = t - \frac{1}{3}t^3$ about x-axis.

[GKP, 1996, 99]

Solution. The equations of the curve are

$$x = t^2,\ y = t - \frac{1}{3}t^3.$$

$$\therefore \quad \frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 1 - t^2.$$

$$\text{Hence } \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]}$$

$$= \sqrt{[4t^2 + (1-t^2)^2]}$$

$$= \sqrt{[1+t^2]^2} = 1+t^2.$$

For the loop t varies from 0 to $\sqrt{3}$.

∴ The required surface

$$= 2\pi \int_{t=0}^{\sqrt{3}} y\, ds = 2\pi \int_{t=0}^{\sqrt{3}} y \frac{ds}{dt}\, dt$$

$$= 2\pi \int_0^{\sqrt{3}} \left(t - \frac{1}{3}t^3\right)(1 + t^2)\, dt$$

$$= \frac{2\pi}{3} \int_0^{\sqrt{3}} \left(3t - t^3\right)(1 + t^2)\, dt$$

$$= \frac{2\pi}{3} \int_0^{\sqrt{3}} (3t - t^3 + 3t^3 - t^5)\, dt$$

$$= \frac{2\pi}{3} \int_0^{\sqrt{3}} (3t + 2t^3 - t^5)\, dt$$

$$= \frac{2\pi}{3} \left[\frac{3}{2}t^2 + \frac{1}{2}t^4 - \frac{t^6}{6}\right]_0^{\sqrt{3}}$$

$$= \frac{2\pi}{3} \left[\frac{9}{2} + \frac{9}{2} - \frac{9}{2}\right]$$

$$= 3\pi.$$

Example 9. Prove that the surface of the solid generated by the revolution of tractrix $x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{t}{2}$, $y = a \sin t$, about the asymptote is equal to the surface of a sphere of radius a. **[GKP, 1997]**

Solution. The given curve is

$$x = a\cos t + \frac{1}{2}a\log\tan^2\frac{t}{2},$$

$$y = a\sin t.$$

The shape of the curve is as shown in the figure.

Here

$$\frac{dx}{dt} = -a\sin t + \frac{1}{2}a\frac{1}{\tan^2\frac{t}{2}}2\tan\frac{t}{2}\sec^2\frac{t}{2}\frac{1}{2}$$

$$= -a\sin t + \frac{1}{2}\frac{a}{\sin\frac{t}{2}\cos\frac{t}{2}}$$

$$= -a\sin t + \frac{a}{\sin t}$$

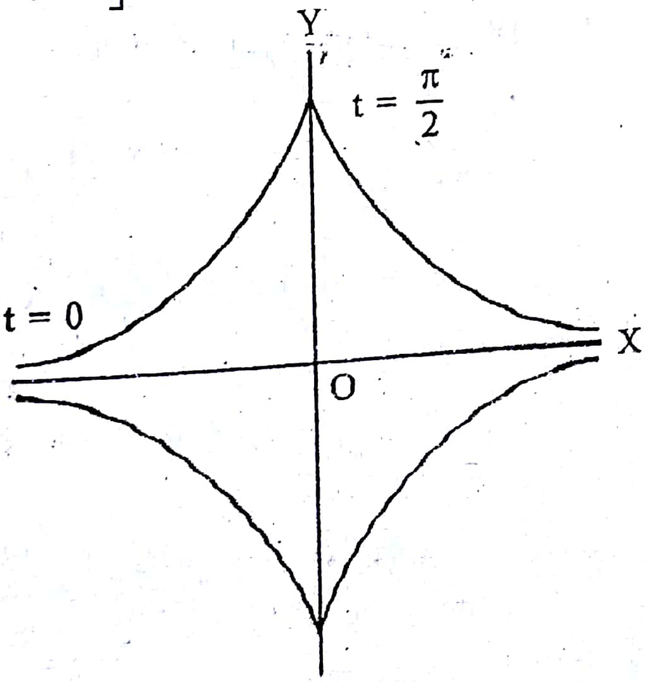
$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t},$$

and $\frac{dy}{dt} = a\cos t.$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = a\sqrt{\left[\frac{\cos^4 t}{\sin^2 t} + \cos^2 t\right]}$$

$$= a\cos t\sqrt{\left[\frac{\cos^2 t + \sin^2 t}{\sin^2 t}\right]}$$

$$= \frac{a\cos t}{\sin t}.$$



The asymptote of the curve is x-axis.

$\therefore$ The required surface

= 2 × Area of surface generated by the portion of curve in a quadrant

$$= 2 \times 2\pi\int_{t=0}^{\pi/2} y\,ds$$

$$= 4\pi\int_{t=0}^{\pi/2} a\sin t\frac{ds}{dt}\,dt$$