

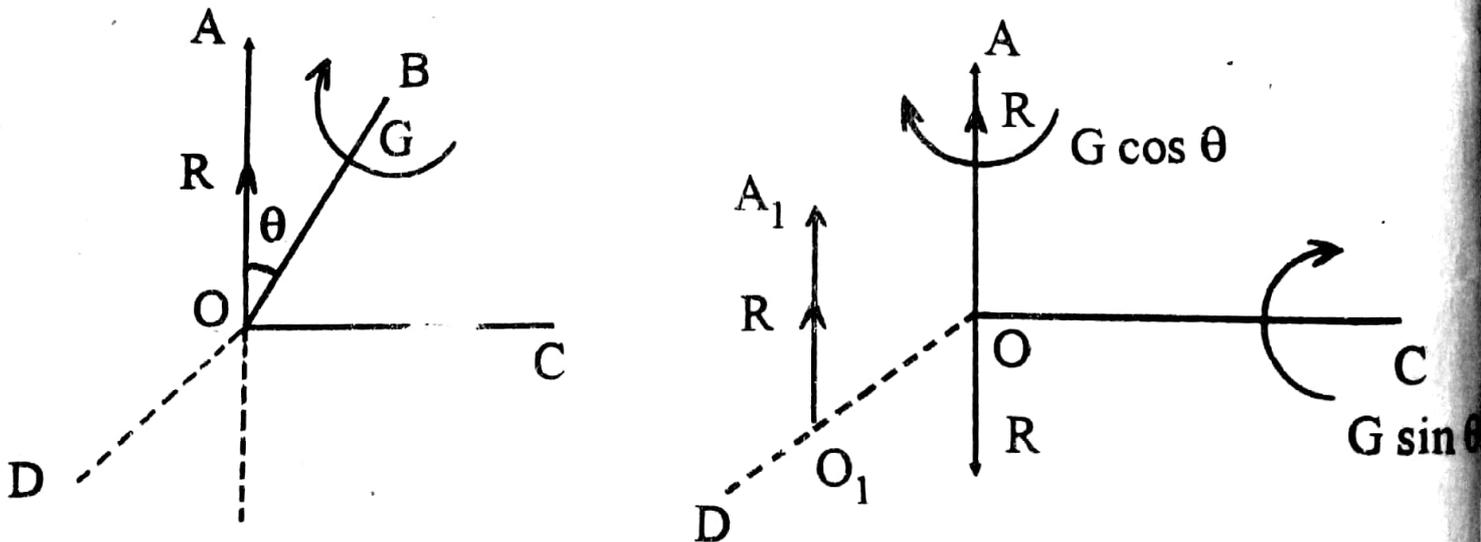
# 2

## Poinsot's Central Axis, Wrenches, Nul Lines and Planes

### 1. Poinsot's Central axis

We have seen in the preceding chapter that any system of forces acting on a rigid body can be reduced to a force  $R$  acting at any point  $O$  and a couple of moment  $G$  about a line through  $O$ .

Now, we will show, that any system of forces acting on a rigid body can be reduced to a single force together with a couple whose axis is along the direction of the force.



Let  $OA$  be the direction of  $R$ ,  $OB$  the axis of the couple  $G$ , and  $\angle AOB = \theta$ . The direction cosines of  $OA$  and  $OB$  are given by  $X/R$ ,  $Y/R$ ,  $Z/R$  and  $L/G$ ,  $M/G$ ,  $N/G$  respectively. Thus, the angle  $\theta$  is given by

$$\cos \theta = \frac{XL + YM + ZN}{RG}$$

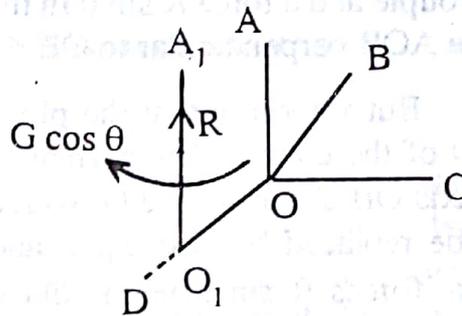
In the plane AOB, draw OC perpendicular to OA, and draw OD perpendicular to the plane AOC. The couple G about OB as axis is equivalent to a couple  $G \cos \theta$  about OA as axis and a couple  $G \sin \theta$  about OC as axis, which acts in the plane AOD, and may therefore be replaced by any two equal unlike parallel forces of moment  $G \sin \theta$ . Choose for one of these two forces a force R at O in the direction opposite to OA. Then the other force must be equal to R acting parallel to OA at a point  $O_1$  in OD, such that

$$R \cdot OO_1 = G \sin \theta, \quad \text{or} \quad OO_1 = \frac{G \sin \theta}{R}$$

The forces at O balance each other and the axis of the couple  $G \cos \theta$  may be transferred from OA to  $O_1A_1$ .

Thus, finally we have a force R along  $O_1A_1$  and a couple of moment  $G \cos \theta$  about  $O_1A_1$ , as axis.

This axis  $O_1A_1$  is called **Poinsot's Central Axis**.



**Def.** Any system of forces acting on a rigid body can be reduced to a single force passing through a point O and a couple whose axis passes through O. This single force R and couple G can be changed in such a way that the axis of the couple is along the direction of the force, then this axis is called **central axis**.

## 2. Some definitions

**Wrench.** A single force R together with a couple K whose axis coincides with the direction of the force are taken together, called a **Wrench**.

**Pitch.** The ratio  $K/R$ , viz the moment of the couple divided by the force, is called the **Pitch** and is a linear magnitude.

When the pitch is zero, the wrench reduces to a single force. When the pitch is infinite, the wrench becomes a couple only.

**Intensity of the wrench.** The single force R is often called the **intensity of the wrench**.

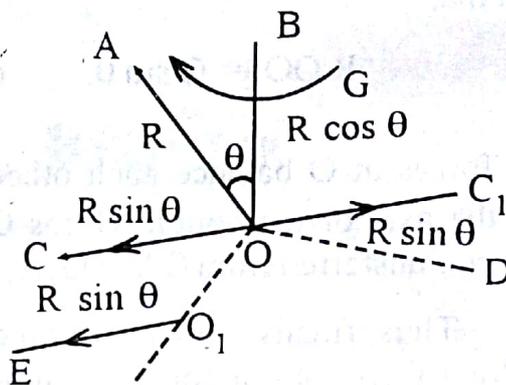
**Screw.** The straight line along which the single force acts when considered together with the pitch is called a **Screw**, so that a Screw is a definite straight line associated with a definite pitch.

### 3. Condition for a given system of forces to compound into a single force

We know that a number of forces acting at different points of a rigid body can be reduced to a single force  $R$  acting at any arbitrarily chosen point  $O$  and a single couple  $G$ .

Let  $OA$  be the direction of  $R$  and  $OB$  be the axis of the couple  $G$ .  $\theta$  be the angle between the axis  $OB$  of  $G$ , then  $R$  is equivalent to a force  $R \cos \theta$  along the axis  $OB$  of the couple and a force  $R \sin \theta$  in the plane  $AOB$  perpendicular to  $OB$ .

But we know that the plane  $COD$  of the couple  $G$  is normal to the axis  $OB$  of the couple  $G$ , which can be replaced by two equal and unlike forces  $R \sin \theta$  in the plane  $COD$  of the couple, one can be taken opposite to  $OC$  in the direction  $OC_1$  and another in the direction  $O_1E$ , which is parallel to the force along  $OC$  and acting through  $O_1$ , where  $OO_1 = (R \sin \theta)/G$ . The forces  $R \sin \theta$  at  $O$  balance each other. So, we are left with one force  $R \sin \theta$  along  $O_1E$ , which does not intersect the force  $R \cos \theta$  at  $O$ . Thus, the force  $R \cos \theta$  acting at  $O$  and  $R \sin \theta$  acting at  $O_1$ , can not be combined in a single force.



But, if  $R \cos \theta = 0$  i.e., if  $\cos \theta = 0$ , then we are left with a single force  $R \sin \theta$ . Hence,  $\theta$  must be  $90^\circ$ . The angle  $\theta$  between the straight lines  $OA$  and  $OB$ , whose direction cosines are  $X/R, Y/R, Z/R$  and  $L/G, M/G, N/G$  are given by

$$\cos \theta = \frac{XL + YM + ZN}{RG}$$

If  $\theta = 90^\circ$ , then  $XL + YM + ZN = 0$ , which is the condition for the system of forces to reduce to a single force.

### 4. Invariants

For any given system of forces, the quantities  $X^2 + Y^2 + Z^2$  and  $LX + MY + NZ$  are invariants, whatever be the origin, or base point and axes.

We know that  $X^2 + Y^2 + Z^2$  is the square of the resultant force  $R$  corresponding to the central axis, which is unique for a system of forces,

therefore,  $X^2 + Y^2 + Z^2$  is an invariant.

We know that the direction cosines of the resultant force are  $X/R$ ,  $Y/R$ ,  $Z/R$  and that of the axis of the resultant couple are  $L/G$ ,  $M/G$ ,  $N/G$  then the cosine of the angle  $\theta$  between the resultant force and the axis of the resultant couple is given by

$$\cos \theta = \frac{XL + YM + ZN}{RG}$$

or

$$XL + MY + NZ = RG \cos \theta = RK$$

where  $K$  is the moment of the couple about the central axis.

Hence,  $I \equiv LX + MY + NZ$  is an invariant.

### 5. Equation of a central axis

Let  $Q$  be any point having coordinates  $(\xi, \eta, \zeta)$  referred to the axes  $Ox, Oy, Oz$ . The moment about a line through  $Q$  parallel to  $Ox$  is obtained by putting  $x_1 - \xi, y_1 - \eta, z_1 - \zeta$  instead of  $x_1, y_1, z_1$  in the expression for the moments  $L, M, N$  as the coordinates of any point  $P(x_1, y_1, z_1)$  are changed to  $(x_1 - \xi, y_1 - \eta, z_1 - \zeta)$  due to the shifting of the axes through  $Q$  parallel to the original axes.

Hence, the moment about a line through  $Q$  parallel to  $Ox$  axis is

$$\begin{aligned} \Sigma\{(y_1 - \eta)Z_1 - (z_1 - \zeta)Y_1\} &= \Sigma(y_1 Z_1 - z_1 Y_1) - \eta \Sigma Z_1 + \zeta \Sigma Y_1 \\ &= L - \eta Z + \zeta Y = L' \end{aligned}$$

Similarly, the moments about the lines through  $Q$  parallel to  $Oy$  and  $Oz$  axes are

$$\Sigma\{(z_1 - \zeta)X_1 - (x_1 - \xi)Z_1\} = M - \zeta X + \xi Z = M'$$

and

$$\Sigma\{(x_1 - \xi)Y_1 - (y_1 - \eta)X_1\} = N - \xi Y + \eta X = N'$$

Also, the components of the resultant force are the same for all points such as  $Q$ , and are thus  $X, Y$  and  $Z$ .

If  $Q$  be a point on the central axis, the direction cosines  $(L'/G', M'/G', N'/G')$  of the axis of the couple corresponding to  $Q$  are proportional to the direction cosines  $(X/R, Y/R, Z/R)$  of the resultant force. Hence

$$\frac{L'}{X} = \frac{M'}{Y} = \frac{N'}{Z}$$

or

$$\frac{L - \eta Z - \zeta Y}{X} = \frac{M - \zeta X + \xi Z}{Y} = \frac{N - \xi Y + \eta X}{Z}$$

These ratios are also equal to

$$= \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = \frac{K}{R}$$

Hence, the equation of the locus of the point  $(\xi, \eta, \zeta)$  i.e., the required equation of the central axis, is

$$\frac{L - yZ - zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{K}{R}$$

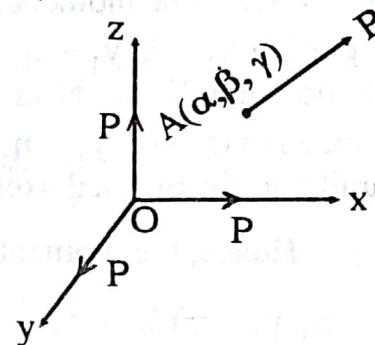
= the pitch  $p$  of the wrench.

**Exp. (5.1)** Equal forces act along the axes and along the straight line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n},$$

find the equations of the central axis of the system. (Gkp.'03, 05)

**Sol.** Let the three equal forces  $P$  be acting along the axes  $Ox, Oy, Oz$  and a force  $P$  be acting along the given line which passes through the point  $A(\alpha, \beta, \gamma)$  and has direction cosines  $l, m, n$ . So the components of this force along the axes are  $Pl, Pm, Pn$ . Thus, the components of the resultant force are



$$X = P + Pl = P(1 + l), \quad Y = P + Pm = P(1 + m), \quad Z = P + Pn = P(1 + n)$$

In order to find the couple of moments  $L, M, N$ , we shall write the coordinates of points of application in one row and the components in another row, then we can find  $L, M, N$  easily, that is,

0,	0,	0	0,	0,	0	0,	0,	0	$\alpha,$	$\beta,$	$\gamma$
P,	0,	0	0,	P,	0	0,	0,	P	$Pl,$	$Pm,$	$Pn$

$$L = 0 - 0 + 0 - 0 \cdot P + 0 \cdot P - 0 + \beta \cdot Pn - \gamma \cdot Pm = (\beta n - \gamma m) P$$

$$M = 0 \cdot P - 0 + 0 - 0 + 0 - 0 \cdot P + \gamma \cdot Pl - \alpha \cdot Pn = (\gamma l - \alpha n) P$$

$$N = 0 - 0 \cdot P + 0 \cdot P - 0 + 0 - 0 + \alpha \cdot Pm - \beta \cdot Pl = (\alpha m - \beta l) P$$

Substituting these values in the equation

$$\frac{L - yZ - zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

we have

$$\frac{(\beta n - \gamma m)P - yP(1+n) + zP(1+m)}{P(1+l)} = \frac{(\gamma l - \alpha n)P - zP(1+l) + xP(1+n)}{P(1+m)}$$

$$= \frac{(\alpha m - \beta l)P - xP(1+m) + yP(1+l)}{P(1+n)}$$

or

$$\frac{\beta n - \gamma m - y(1+n) + z(1+m)}{(1+l)} = \frac{\gamma l - \alpha n - z(1+l) + x(1+n)}{(1+m)}$$

$$= \frac{\alpha m - \beta l - x(1+m) + y(1+l)}{(1+n)}$$

**Exp. (5.2)** Forces  $X, Y, Z$  act along the three lines given by the equations  $y = 0, z = c; z = 0, x = a; x = 0, y = b$ ; prove that the pitch of the equivalent wrench is

$$\frac{(aYZ + bZX + cXY)}{(X^2 + Y^2 + Z^2)}$$

If the wrench reduces to a single force, show that the line of action of the force lies on the hyperboloid  $(x - a)(y - b)(z - c) = xyz$ .

**Sol.** These three lines can be written as

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-c}{0}, \quad \frac{x-a}{0} = \frac{y-0}{1} = \frac{z-0}{0}, \quad \frac{x-0}{0} = \frac{y-b}{0} = \frac{z-0}{1}$$

So, the force  $X$  is acting along a line parallel to  $x$ -axis from the point  $(0, 0, c)$ , the force  $Y$  is acting along a line parallel to  $y$ -axis from the point  $(a, 0, 0)$  and the force  $Z$  is acting along the line parallel to  $z$ -axis from the point  $(0, b, 0)$ .

The three component couples are

$$\begin{matrix} 0, & 0, & c & a, & 0, & 0 & 0, & b, & 0 \\ X, & 0, & 0 & 0, & Y, & 0 & 0, & 0, & Z \end{matrix}$$

$$L = 0 - 0 \cdot c + 0 - 0 \cdot Y + b \cdot Z - 0 = bZ$$

$$M = cX - 0 + 0 - a \cdot 0 + 0 - 0 \cdot Z = cX$$

$$N = 0 - 0 \cdot X + aY - 0 + 0 - 0 \cdot b = aY$$

The pitch  $p$  of the resultant wrench is given by  $\frac{LX + MY + NZ}{X^2 + Y^2 + Z^2}$ . On substituting the above values in this expression, we have

$$p = \frac{bZX + cXY + aYZ}{X^2 + Y^2 + Z^2}$$

If the wrench reduces to single force (i.e.,  $LX + MY + NZ = 0$ ), the equations of the central axis become

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = 0.$$

On substituting the values of L, M, N in the above relations, we have

$$(i) \quad bZ - yZ + zY = 0, \quad (ii) \quad cX - zX + xZ = 0, \quad (iii) \quad aY - xY + yX = 0$$

From (i), (ii) and (iii), we have

$$(iv) \quad \frac{Y}{Z} = \frac{y-b}{z}, \quad (v) \quad \frac{Z}{X} = \frac{z-c}{x}, \quad (vi) \quad \frac{X}{Y} = \frac{x-a}{y}$$

In order to find the equation of the surface on which the line of action of the force lies, we have to eliminate X, Y, Z.

From (iv) and (v), we have

$$(vii) \quad \frac{Y}{X} = \frac{Y}{Z} \cdot \frac{Z}{X} = \frac{y-b}{z} \cdot \frac{z-c}{x}$$

On multiplying (vi) and (vii), we have

$$\frac{X}{Y} \cdot \frac{Y}{X} = \frac{x-a}{y} \cdot \frac{y-b}{z} \cdot \frac{z-c}{x}$$

$$\text{or} \quad xyz = (x-a)(y-b)(z-c).$$

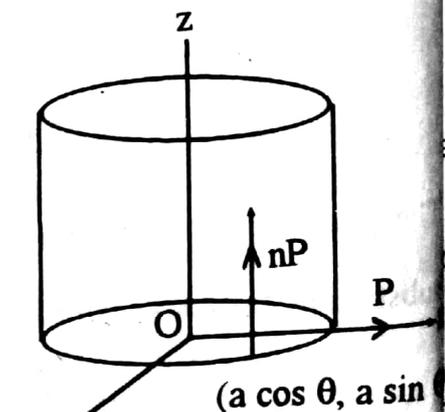
This is the required equation of hyperboloid:

**Exp. (5.3)** A force P acts along the axis of x and another force nP along a generator of the cylinder  $x^2 + y^2 = a^2$ , show that the central axis lies on the cylinder

$$n^2 (nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2. \quad (\text{Gkp.'06, 09})$$

**Sol.** The two forces P and nP are acting along x - axis and the generator to the given cylinder  $x^2 + y^2 = a^2$ , respectively. The generators of the cylinder are parallel to z - axis, and let  $(a \cos \theta, a \sin \theta, 0)$  be a point on the cylinder, then the equation of the generator can be taken as

$$\frac{x - a \cos \theta}{0} = \frac{y - a \sin \theta}{0} = \frac{z - 0}{1}$$



The components of the resultant force are  $P, 0, nP$ .

Here, we have

$$\begin{array}{ccc} 0, & 0, & 0 \\ P, & 0, & 0 \end{array} \quad \begin{array}{ccc} a \cos \theta, & a \sin \theta, & 0 \\ 0, & 0, & nP \end{array}$$

Hence,

$$L = 0 - 0 + nPa \sin \theta - 0 = a n P \sin \theta$$

$$M = 0.P - 0 + 0 - a n P \cos \theta = -a n P \cos \theta$$

$$N = 0 - 0.P + 0. a \cos \theta - 0. a \sin \theta = 0 \quad \text{and } X = P, Y = 0, Z = nP.$$

Substituting these values in the equations, we find the equation of central axis.

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z},$$

we have

$$\frac{anP \sin \theta - ynP}{P} = \frac{-anP \cos \theta - zP + xnP}{0} = \frac{0 - x0 + yP}{nP}$$

$$\text{or } \frac{n(a \sin \theta - y)}{1} = \frac{-an \cos \theta - z + xn}{0} = \frac{y}{n}$$

Therefore

$$xn - z = an \cos \theta$$

or

$$(i) \quad an^2 \cos \theta = n(xn - z)$$

on taking first and third ratios, we have

$$n^2(a \sin \theta - y) = y$$

or

$$(ii) \quad n^2 a \sin \theta = y(1 + n^2).$$

On squaring and adding (i) and (ii), we have

$$a^2 n^4 = y^2(1 + n^2)^2 + n^2(nx - z)^2$$

**Exp. (5.4)** Three forces act along the straight lines  $x = 0, y - z = a; y = 0, z - x = a; z = 0, x - y = a$ . Show that they cannot reduce to a couple. (Gkp.'02)

Prove also that the system reduces to a single force its line of action must lie on the surface  $x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = a^2$ .

**Sol.** Let the three forces  $P, Q, R$  be acting along the lines  $x = 0, y - z = a; y = 0, z - x = a; z = 0, x - y = a$  respectively. The coordinates of the points, which lie on these lines can be taken as  $(0, a, 0), (0, 0, a),$

(a, 0, 0) from these points the forces P, Q, R are acting. The direction cosines of these lines are 0,  $1/\sqrt{2}$ ,  $1/\sqrt{2}$ ;  $1/\sqrt{2}$ , 0,  $1/\sqrt{2}$ ;  $1/\sqrt{2}$ ,  $1/\sqrt{2}$ , 0. respectively. The components of these forces along the axes are 0,  $P/\sqrt{2}$ ,  $P/\sqrt{2}$ ;  $Q/\sqrt{2}$ ,  $Q/\sqrt{2}$ , 0;  $R/\sqrt{2}$ ,  $R/\sqrt{2}$ , 0.

Thus, the components of the resultant force along the axes are

$$X = 0 + \frac{Q}{\sqrt{2}} + \frac{R}{\sqrt{2}} = \frac{Q+R}{\sqrt{2}}$$

$$Y = \frac{P}{\sqrt{2}} + 0 + \frac{R}{\sqrt{2}} = \frac{P+R}{\sqrt{2}}$$

$$Z = \frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} + 0 = \frac{P+Q}{\sqrt{2}}$$

Then the square of the resultant force is given  $\{(Q+R)^2 + (P+R)^2 + (P+Q)^2\}/2$  which can not be zero unless P, Q, R are zero. Hence, the system cannot reduce to a single couple.

On writing the coordinates of point of action and the components of the forces in the following manner, we can easily calculate the couple moments about the axes.

$$\begin{array}{ccc} 0, & a, & 0 \\ 0, & \frac{P}{\sqrt{2}}, & \frac{P}{\sqrt{2}} \end{array} \quad \begin{array}{ccc} 0, & 0, & a \\ \frac{Q}{\sqrt{2}}, & 0 & \frac{Q}{\sqrt{2}} \end{array} \quad \begin{array}{ccc} a, & 0, & 0 \\ \frac{R}{\sqrt{2}}, & \frac{R}{\sqrt{2}}, & 0 \end{array}$$

Thus,

$$L = aP/\sqrt{2} - 0.P/\sqrt{2} + 0.Q/\sqrt{2} - 0.a + 0 - 0.R/\sqrt{2} = aP/\sqrt{2}$$

$$M = 0 - 0.P/\sqrt{2} + aQ/\sqrt{2} - 0.Q/\sqrt{2} + 0.R/\sqrt{2} - a.0 = aQ/\sqrt{2}$$

$$N = 0.P/\sqrt{2} - 0.a + 0 - 0.Q/\sqrt{2} + aR/\sqrt{2} - 0.R/\sqrt{2} = aR/\sqrt{2}$$

Substituting these values in the equations

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

we have

$$\frac{aP/\sqrt{2} - y(P+Q)/\sqrt{2} + z(P+R)/\sqrt{2}}{(Q+R)/\sqrt{2}}$$

$$= \frac{aQ/\sqrt{2} - z(Q+R)/\sqrt{2} + x(P+Q)/\sqrt{2}}{(P+R)/\sqrt{2}}$$

$$= \frac{aR/\sqrt{2} - x(P+R)/\sqrt{2} + y(Q+R)/\sqrt{2}}{(P+Q)/\sqrt{2}}$$

$$\begin{aligned} \text{or } \frac{(a-y+z)P - yQ + zR}{Q+R} &= \frac{xP + (a-z+x)Q - zR}{P+R} \\ &= \frac{-xP + yQ + (a-x+y)R}{P+Q} = 0. \end{aligned}$$

Since the system reduces to a single force that is, the moment of the couple vanishes. Thus,

$$\begin{aligned} (a-y+z)P - yQ + zR &= 0 \\ xP + (a-z+x)Q - zR &= 0 \\ -xP + yQ + (a-x+y)R &= 0. \end{aligned}$$

Eliminating P, Q, R from the above equations, we have

$$\begin{vmatrix} a-y+z & -y & z \\ x & a-z+x & -z \\ -x & y & a-x+y \end{vmatrix} = 0$$

$$\text{or } \begin{vmatrix} a-y+z+x & a-y-z+x & 0 \\ 0 & a-z+x+y & a-x+y-z \\ -x & y & a-x+y \end{vmatrix} = 0$$

(R<sub>1</sub> + R<sub>2</sub>, R<sub>2</sub> + R<sub>3</sub>)

$$\text{or } \begin{vmatrix} a-y+z+x & -2z & 0 \\ 0 & a-z+x+y & a-x+y-z \\ -x & x+y & a-x+y \end{vmatrix} = 0$$

(C<sub>2</sub> - C<sub>1</sub>)

$$\text{or } \begin{vmatrix} a+x-y+z & -2z & 0 \\ 0 & 2x & a-x+y-z \\ -x & 2x-a & a-x+y \end{vmatrix} = 0$$

(C<sub>2</sub> - C<sub>3</sub>)

$$\text{or } \begin{vmatrix} a+x-y+z & -2z & 0 \\ 0 & 2x & a-x+y-z \\ -x & -a & z \end{vmatrix} = 0$$

(R<sub>3</sub> - R<sub>2</sub>)

Expanding along the first row, we have

$$(a+x-y+z) \{ 2xz + a(a-x+y-z) \} - 2z \{ -x(a-x+y-z) \} = 0$$

$$\text{or } 2axz + 2xz(x-y+z) + a^2(a+x-y+z) - a(x-y+z)(a+x-y+z)$$

$$+ 2xz(a - x + y - z) = 0$$

$$\text{or } 2axz + 2xz(x - y + z) + a^3 + a^2(x - y + z) - a^2(x - y + z) - a(x - y + z)^2 + 2axz - 2xz(x - y + z) = 0,$$

$$\text{or } a^3 - a(x - y + z)^2 + 4axz = 0$$

$$\text{or } (x - y + z)^2 - 4xz = a^2$$

$$\text{or } x^2 + y^2 + z^2 - 2xy - 2yz - 2xz = a^2.$$

**Exp. (5.5)** A single force is equivalent to component forces  $X$ ,  $Y$ , and  $Z$  along the axes of coordinates and to couples  $L$ ,  $M$ ,  $N$  about the axes. Show that the magnitude of the single force is  $(X^2 + Y^2 + Z^2)^{1/2}$  and that the equation to its line of action is

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1. \quad (\text{Gkp. } 10)$$

**Sol.** A single force is equivalent to component forces  $X$ ,  $Y$ ,  $Z$  along the coordinate axes; hence the magnitude of the resultant force will be  $(X^2 + Y^2 + Z^2)^{1/2}$ . But it is given that the system reduces to a single force. Therefore,

$$(i) \quad LX + MY + NZ = 0.$$

Now, the equations of the central axis of the system are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = 0$$

(due to (i))

Thus,

$$L - yZ + zY = 0, \quad M - zX + xZ = 0, \quad N - xY + yX = 0$$

which gives

$$\frac{yZ - zY}{L} = 1, \quad \frac{zX - xZ}{M} = 1, \quad \frac{xY - yX}{N} = 1$$

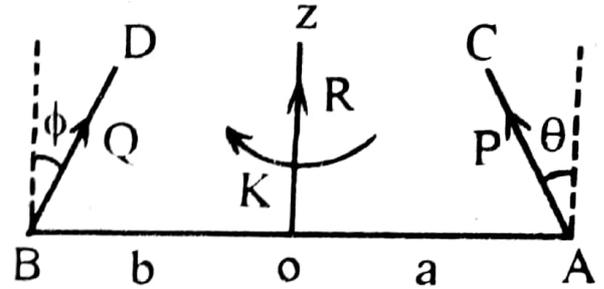
and hence the equation of the line of action of the single resultant are

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1.$$

## 6. Resolution of a system of forces in two forces

Let  $Oz$  be the central axis of a given system whose resultant force  $R$  acts along  $Oz$  and resultant couple  $K$  has  $Oz$  as its axis.

Take any two points  $A$  and  $B$  on a line through  $O$  perpendicular to  $Oz$  on opposite sides of  $O$ , such that  $OA = a$  and  $OB = b$ .



Assume that the given system is equivalent to a force  $P$  acting through  $A$  in a plane perpendicular to  $OA$  at an angle  $\theta$  with  $Oz$  and a force  $Q$  acting through  $B$  in a plane perpendicular to  $OB$  at an angle  $\phi$  with  $Oz$ . Let  $\theta$  and  $\phi$  be measured positively in opposite direction.

Then, according to our assumption, the resultant of  $P$  and  $Q$  along  $Oz$  and perpendicular to it must be  $R$  and zero respectively. So,

$$(6.1) \quad R = P \cos \theta + Q \cos \phi$$

and

$$(6.2) \quad 0 = P \sin \theta - Q \sin \phi.$$

Similarly, the resultant couple about these two lines must be  $K$  and zero.

$$(6.3) \quad K = P \sin \theta \cdot a + Q \sin \phi \cdot b$$

and

$$(6.4) \quad 0 = P \cos \theta \cdot a - Q \cos \phi \cdot b.$$

On writing equation (6.4) and (6.1) in the following form,

$$a P \cos \theta - b Q \cos \phi + 0 = 0$$

$$P \cos \theta + Q \cos \phi - R = 0$$

we have

$$\frac{P \cos \theta}{bR} = \frac{Q \cos \phi}{aR} = \frac{1}{a+b}$$

or

$$(6.5) \quad \frac{P \cos \theta}{b} = \frac{Q \cos \phi}{a} = \frac{R}{a+b}$$

Similarly, from the equations (6.2) and (6.3), we have

$$(6.6) \quad P \sin \theta = Q \sin \phi = \frac{K}{a+b}$$

From (6.5), we have

$$(6.7) \quad P \cos \theta = \frac{bR}{a+b}, \quad Q \cos \phi = \frac{aR}{a+b}$$

On squaring and adding, we have

$$P^2 = \frac{K^2 + b^2 R^2}{(a+b)^2}, \quad Q^2 = \frac{K^2 + a^2 R^2}{(a+b)^2}$$

and

$$\tan \theta = \frac{K}{bR}, \quad \tan \phi = \frac{K}{aR}$$

For a set of values of  $a$  and  $b$ , we can obtain real values of  $P$ ,  $Q$ ,  $\theta$  and  $\phi$ . Thus, the forces  $P$  and  $Q$  can be obtained in an infinite number of ways, by choosing  $a$  and  $b$  in infinite number of ways.

From (6.6) and (6.7), we have

$$KR a = (a+b)^2 P Q \sin \theta \cos \phi, \quad \text{and} \quad KR b = (a+b)^2 P Q \cos \theta \sin \phi.$$

On adding these two, we have

$$(6.8) \quad KR = P Q (a+b) \sin (\theta + \phi)$$

Let  $AC$ ,  $BD$  represent  $P$ ,  $Q$  in magnitude and direction.

The volume of the tetrahedron  $ACBD$

$$= \frac{1}{3} \times \text{Area of the } \Delta ABC \times \text{perpendicular from } D \text{ upon the } \Delta ABC$$

$$= \frac{1}{3} \times \frac{1}{2} AB \cdot AC \times BD \sin (\theta + \phi) = \frac{1}{6} P Q (a+b) \sin (\theta + \phi)$$

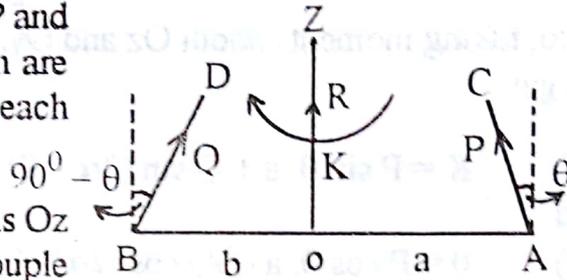
$$= \frac{1}{6} KR \quad (\text{by eq. (6.8)})$$

and it is therefore constant.

**Exp. (6.1)** If P and Q be two non-intersecting forces whose directions are perpendicular, show that the distances of the central axis from their lines of action are as  $Q^2$  to  $P^2$ . (Purv.'90)

**Sol.** Let the two forces P and Q be acting along AC, BD which are perpendicular and do not intersect each other.

Suppose their central axis is Oz and resultant is R and resultant couple is K. The distance between O and A is a and between O and B is b.



Now, resolving P and Q along Oz and perpendicular to it, we have

(i)  $R = P \cos \theta + Q \sin \theta$   
 (ii)  $0 = P \sin \theta - Q \cos \theta$

and for couple K, we have

(iii)  $K = aP \sin \theta + bQ \cos \theta$   
 (iv)  $0 = aP \cos \theta - bQ \sin \theta$ .

From (ii) and (iv), we have

$\frac{P \sin \theta}{Q \cos \theta} = 1$  or  $\frac{\sin \theta}{\cos \theta} = \frac{Q}{P}$ ,  
 and  $\frac{aP \cos \theta}{bQ \sin \theta} = 1$  or  $\frac{\sin \theta}{\cos \theta} = \frac{aP}{bQ}$ ,

then, from these two, we have

$\frac{Q}{P} = \frac{aP}{bQ}$  or  $\frac{a}{b} = \frac{Q^2}{P^2}$ .

**Exp. (6.2)** Show that the minimum distance between two forces, which are equivalent to a given system (K,R) and which are inclined at a given angle  $2\alpha$ , is  $\frac{2K}{R} \cot \alpha$ , and that the forces are then each equal to  $\frac{1}{2} R \sec \alpha$ . (Gkp.'01)

**Sol.** The system is equivalent to a single force R along Oz and a single couple K about Oz.

Resolving the forces along the axes z and y, we get

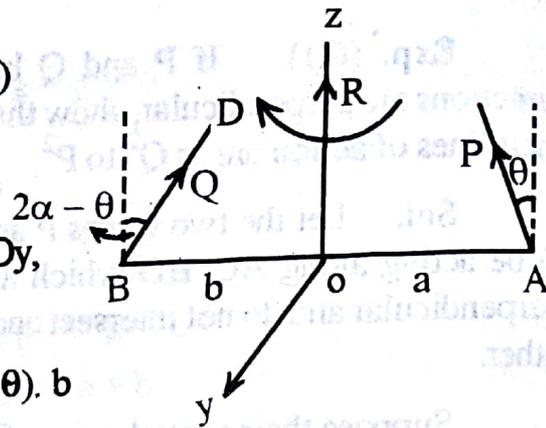
(i)  $R = P \cos \theta + Q \cos (2\alpha - \theta)$   
and

(ii)  $0 = P \sin \theta - Q \sin (2\alpha - \theta)$

Also, taking moments about Oz and Oy,  
we get

(iii)  $K = P \sin \theta \cdot a + Q \sin (2\alpha - \theta) \cdot b$   
and

(iv)  $0 = P \cos \theta \cdot a - bQ \cos (2\alpha - \theta)$



From (i) and (iv), we have

(v)  $\frac{P \cos \theta}{a} = \frac{Q \cos (2\alpha - \theta)}{b} = \frac{R}{a + b}$

and from (ii) and (iii),

(vi)  $P \sin \theta = Q \sin (2\alpha - \theta) = \frac{K}{a + b}$

From (v) and (vi), we have

$$a \tan \theta = \frac{K}{R}, \quad a = \frac{K}{R} \cot \theta$$

$$b \tan (2\alpha - \theta) = \frac{K}{R}, \quad b = \frac{K}{R} \cot (2\alpha - \theta)$$

Then,  $a + b = \frac{K}{R} \{ \cot \theta + \cot (2\alpha - \theta) \}$

$$= \frac{K}{R} \left\{ \frac{\cos \theta \sin (2\alpha - \theta) + \sin \theta \cos (2\alpha - \theta)}{\sin \theta \sin (2\alpha - \theta)} \right\}$$

$$= \frac{2K}{R} \cdot \frac{\sin 2\alpha}{2 \sin \theta \sin (2\alpha - \theta)}$$

(vii)  $= \frac{2K}{R} \cdot \frac{\sin 2\alpha}{\cos 2(\alpha - \theta) - \cos 2\alpha}$

Except  $\cos 2(\alpha - \theta)$  all are given, so we have to make this maximum for the quantity to be minimum. So,

$$\cos 2(\alpha - \theta) = \cos 0 \quad \text{or} \quad 2(\alpha - \theta) = 0, \quad \alpha = \theta.$$

Substituting  $\theta = \alpha$ , in (vii), we have

$$a + b = \frac{2K}{R} \cdot \frac{\sin 2\alpha}{1 - \cos 2\alpha} = \frac{2K}{R} \cdot \frac{\sin 2\alpha}{1 - 1 + 2\sin^2 \alpha} = \frac{2K}{R} \cdot \frac{2 \sin \alpha \cos \alpha}{2 \sin^2 \alpha}$$

$$= \frac{2K}{R} \cot \alpha.$$

From equation (vi), we have

$$P \sin \alpha = Q \sin \alpha = \frac{K.R}{2K \cot \alpha}$$

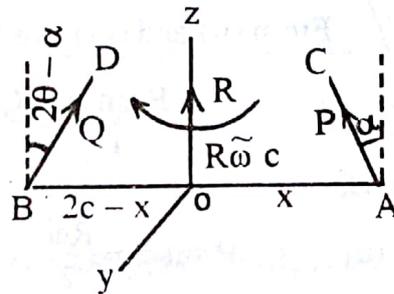
$$P = \frac{R}{2} \cdot \sec \alpha = Q.$$

So, each of the forces are  $\frac{1}{2} R \sec \alpha$ .

**Exp. (6.3)** Show that a wrench, of which the force is  $R$  and the pitch is  $\tilde{\omega}c$  may be replaced by forces inclined at an angle  $2\theta$  to each other, the shortest distance between them being  $2c$ , and that their magnitudes are

$$\frac{R}{2} [\sqrt{(1 + \tilde{\omega} \tan \theta)} \pm \sqrt{(1 - \tilde{\omega} \cot \theta)}].$$

**Sol.** Let the wrench be  $(R, R\tilde{\omega}c)$ , which is equivalent to two forces  $P, Q$  inclined at an angle  $2\theta$  with each other. Let the direction of  $P$  make an angle  $\alpha$  with the axis of  $z$  and that of  $Q$  an angle  $2\theta - \alpha$  with  $Oz$  but in opposite sense.



Now resolving the forces along the axes  $z$  and  $y$ , we get

$$(i) \quad R = P \cos \alpha + Q \cos (2\theta - \alpha)$$

$$(ii) \quad 0 = P \sin \alpha - Q \sin (2\theta - \alpha)$$

Also, taking the moments about  $Oz$  and  $Oy$ , we get

$$(iii) \quad R\tilde{\omega}c = P \sin \alpha \cdot x + Q \sin (2\theta - \alpha) \cdot (2c - x)$$

$$(iv) \quad 0 = P \cos \alpha \cdot x - (2c - x) Q \cos (2\theta - \alpha)$$

The resultant of two forces  $P$  and  $Q$  is

$$(v) \quad R^2 = P^2 + Q^2 + 2PQ \cos 2\theta$$

$$= (P + Q)^2 - 2PQ + 2PQ \cos 2\theta$$

$$= (P + Q)^2 - 2PQ (1 - \cos 2\theta)$$

$$= (P + Q)^2 - 4PQ \sin^2 \theta$$

or

$$(vi) \quad (P + Q)^2 = R^2 + 4PQ \sin^2 \theta$$

Equating (v) can be written as

$$R^2 = (P - Q)^2 + 2PQ(1 + \cos 2\theta) = (P - Q)^2 + 4PQ \cos^2 \theta$$

or

$$(vii) \quad (P - Q)^2 = R^2 - 4PQ \cos^2 \theta$$

From equations (i) and (iv), we have

$$\frac{P \cos \alpha}{2c - x} = \frac{Q \cos (2\theta - \alpha)}{x} = \frac{R}{2c},$$

that is

$$(viii) \quad (a) \quad P \cos \alpha = \frac{R(2c - x)}{2c} \quad (b) \quad Q \cos (2\theta - \alpha) = \frac{Rx}{2c}$$

From (ii) and (iii), we have

$$(ix) \quad \frac{P \sin \alpha}{1} = \frac{Q \sin (2\theta - \alpha)}{1} = \frac{R\tilde{\omega}c}{2c}$$

that is,

$$(x) \quad (a) \quad P \sin \alpha = \frac{R\tilde{\omega}}{2} \quad (b) \quad Q \sin (2\theta - \alpha) = \frac{R\tilde{\omega}}{2}$$

Multiplying (viii)(a) and (x)(b), we have

$$(xi) \quad P \cos \alpha \cdot Q \sin (2\theta - \alpha) = \frac{1}{4} \frac{R^2 \tilde{\omega} (2c - x)}{c}$$

Similarly, from (viii)(b) and (x)(a), we have

$$(xii) \quad P \sin \alpha \cdot Q \cos (2\theta - \alpha) = \frac{1}{4} \frac{R^2 \tilde{\omega} x}{c}$$

Adding (xi) and (xii), we have

$$(xiii) \quad PQ \sin 2\theta = \frac{1}{2} R^2 \tilde{\omega} \quad \text{or} \quad PQ = \frac{R^2 \tilde{\omega}}{2 \sin 2\theta}$$

Substituting from (xiii) in (vi), we have

$$(P + Q)^2 = R^2 + 4 \cdot \frac{R^2 \tilde{\omega} \sin^2 \theta}{2 \sin 2\theta} = R^2 (1 + \tilde{\omega} \tan \theta)$$

or

$$(xiv) \quad P + Q = R \sqrt{1 + \tilde{\omega} \tan \theta}$$

Similarly, from (xiii) and (vii), we have

$$P - Q = R \sqrt{1 - \tilde{\omega} \cot \theta}.$$

Consequently,

$$P = \frac{1}{2} R \{ \sqrt{1 + \tilde{\omega} \tan \theta} + \sqrt{1 - \tilde{\omega} \cot \theta} \}$$

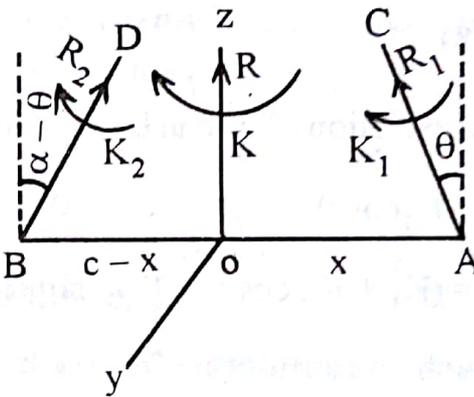
$$Q = \frac{1}{2} R \{ \sqrt{1 + \tilde{\omega} \tan \theta} - \sqrt{1 - \tilde{\omega} \cot \theta} \}.$$

Thus, the two forces are given by

$$\frac{1}{2} R \{ \sqrt{1 + \tilde{\omega} \tan \theta} \pm \sqrt{1 - \tilde{\omega} \cot \theta} \}.$$

### 7. Resultant wrench of two given wrenches

Let AC be the axis of one wrench ( $R_1, K_1$ ) and BD the axis of other wrench ( $R_2, K_2$ ). Let the shortest distance between these two axes be  $AB = c$ , and  $\alpha$  the angle between them.



Suppose that the required central axis Oz is perpendicular to AB, and divides it into parts x and c - x, and is inclined at  $\theta$  to AC, and hence  $\alpha - \theta$  to BD. Further, we assume that the resultant wrench along Oz be ( $R, K$ ).

Now, resolving the forces along the axes z and y, we get

$$(7.1) \quad R = R_1 \cos \theta + R_2 \cos (\alpha - \theta)$$

$$(7.2) \quad 0 = R_1 \sin \theta - R_2 \sin (\alpha - \theta)$$

and equating the couples along the Oz axis and Oy axis, we have

$$(7.3) \quad K = K_1 \cos \theta + K_2 \cos (\alpha - \theta) + R_1 \sin \theta .x + R_2 (c - x) \sin (\theta - \alpha)$$

$$(7.4) \quad 0 = K_1 \sin \theta - K_2 \sin (\alpha - \theta) - R_1 \cos \theta .x + R_2 (c - x) \cos (\theta - \alpha)$$

Substituting from (7.2) in (7.3), we have

$$K = K_1 \cos \theta + K_2 \cos (\alpha - \theta) + R_2 .x \sin (\alpha - \theta) + R_2 (c - x) \sin (\theta - \alpha)$$

$$(7.5) \quad K = K_1 \cos \theta + K_2 \cos (\alpha - \theta) + R_2 c \sin (\alpha - \theta).$$

On substituting from (7.1) in (7.4), we have

$$0 = K_1 \sin \theta - K_2 \sin (\alpha - \theta) + x (R_2 \cos (\alpha - \theta) - R) + R_2 (c - x) \cos (\theta - \alpha)$$

$$(7.6) \quad = K_1 \sin \theta - K_2 \sin (\alpha - \theta) + R_2 c \cos (\theta - \alpha) - Rx$$

Squaring and adding (7.1) and (7.2), we have

$$(7.7) \quad R^2 = R_1^2 + R_2^2 + 2 R_1 R_2 \cos \alpha$$

The equation (7.2) can be written as

$$\begin{aligned} 0 &= R_1 \sin \theta - R_2 (\sin \alpha \cos \theta - \cos \alpha \sin \theta) \\ &= (R_1 + R_2 \cos \alpha) \sin \theta - R_2 \sin \alpha \cos \theta \end{aligned}$$

from which we have

$$(7.8) \quad \frac{\sin \theta}{R_2 \sin \alpha} = \frac{\cos \theta}{R_1 + R_2 \cos \alpha} = \frac{1}{R}$$

The equation (7.5) can be written as

$$\begin{aligned} K &= K_1 \cos \theta + K_2 (\cos \alpha \cos \theta + \sin \alpha \sin \theta) + R_2 c (\sin \alpha \cos \theta - \sin \theta \cos \alpha) \\ &= (K_1 + K_2 \cos \alpha + R_2 c \sin \alpha) \cos \theta + (K_2 \sin \alpha - R_2 c \cos \alpha) \sin \theta \end{aligned}$$

which on substituting from (7.8), we have

$$\begin{aligned} RK &= (K_1 + K_2 \cos \alpha + R_2 c \sin \alpha) (R_1 + R_2 \cos \alpha) \\ &\quad + (K_2 \sin \alpha - R_2 c \cos \alpha) R_2 \sin \alpha \\ &= R_1 K_1 + R_1 K_2 \cos \alpha + R_1 R_2 c \sin \alpha + K_1 R_2 \cos \alpha \\ &\quad + K_2 R_2 \cos^2 \alpha + c R_2^2 \sin \alpha \cos \alpha \\ &\quad + K_2 R_2 \sin^2 \alpha - R_2^2 c \sin \alpha \cos \alpha \\ (7.9) \quad &= R_1 K_1 + R_2 K_2 + (R_1 K_2 + R_2 K_1) \cos \alpha + R_1 R_2 c \sin \alpha \end{aligned}$$

This gives the value of the invariant for the given system of forces.

The equation (7.6) can be written as

$$\begin{aligned} 0 &= K_1 \sin \theta - K_2 \sin \alpha \cos \theta + K_2 \sin \theta \cos \alpha \\ &\quad + R_2 c (\cos \theta \cos \alpha + \sin \theta \sin \alpha) - Rx \\ &= (K_1 + K_2 \cos \alpha + R_2 c \sin \alpha) \sin \theta \\ &\quad + (R_2 c \cos \alpha - K_2 \sin \alpha) \cos \theta - Rx. \end{aligned}$$

On substituting from (7.8), we have

$$R^2 x = (K_1 + K_2 \cos \alpha + R_2 c \sin \alpha) R_2 \sin \alpha + (R_2 c \cos \alpha - K_2 \sin \alpha)$$

$$\begin{aligned} & \times (R_1 + R_2 \cos \alpha) \\ & = K_1 R_2 \sin \alpha + K_2 R_2 \sin \alpha \cos \alpha + R_2^2 c \sin^2 \alpha + R_1 R_2 c \cos \alpha \\ & \quad - R_1 K_2 \sin \alpha + R_2^2 c \cos^2 \alpha - K_2 R_2 \sin \alpha \cos \alpha \\ (7.10) \quad & = (R_2 K_1 - R_1 K_2) \sin \alpha + R_2 c (R_2 + R_1 \cos \alpha) \end{aligned}$$

Equations (7.7), (7.8), (7.9) and (7.10) give R,  $\theta$ , K and x. Thus, we know the required wrench and the position of the axis.

**8. Resultant wrench of two given forces  $R_1$  and  $R_2$  inclined at a given angle  $\alpha$**

This is a particular case of the preceding article where  $K_1 = K_2 = 0$ . Thus, the equations (7.9), (7.10) becomes

$$(8.1) \quad K.R = R_1 R_2 c \sin \alpha$$

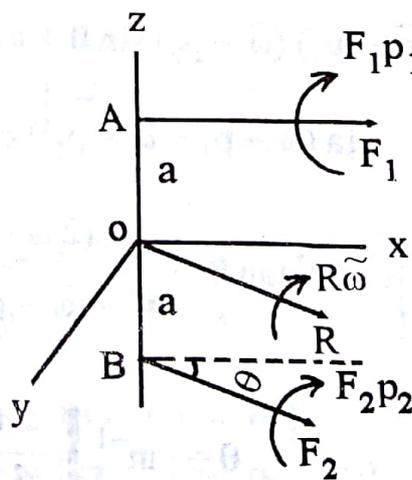
$$(8.2) \quad R^2 x = R_2 c (R_2 + R_1 \cos \alpha)$$

The equations (7.7) and (7.8) remains the same.

**Exp. (8.1)** Two wrenches of pitches  $p_1, p_2$  have axes at a distance  $2a$  from one another. If the resultant wrench is of pitch  $\tilde{\omega}$  and its axis is equidistant from the axes of the component wrenches, show that the angle between them is

$$\tan^{-1} \frac{a(2\tilde{\omega} - p_1 - p_2)}{a^2 - (\tilde{\omega} - p_1)(\tilde{\omega} - p_2)} \quad (\text{Gkp.'00})$$

**Sol.** Let the two wrenches be  $(F_1, p_1 F_1)$ ,  $(F_2, p_2 F_2)$  and the resultant wrench be  $(R, \tilde{\omega} R)$ . Let O be the middle point of the shortest distance  $AB = 2a$ , and let it be z-axis. We have taken the axis of the wrench  $(F_1, p_1 F_1)$  parallel to x-axis and the axis of the wrench  $(F_2, p_2 F_2)$  making an angle  $\theta$  with the x-axis. This lies in the plane parallel to xy plane, AB being the shortest distance. The axis of the resultant wrench  $(R, \tilde{\omega} R)$  passes through the middle point O of the shortest distance and has direction cosines  $l, m, n$ .



Now, resolving the forces along the axes, we have

$$(i) \quad Rl = F_1 + F_2 \cos \theta$$

$$(ii) \quad Rm = F_2 \sin \theta$$

$$(iii) \quad Rn = 0.$$

Also, taking the moments about Ox and Oy, we have

$$(iv) \quad R \tilde{\omega} l = F_1 p_1 + F_2 p_2 \cos \theta + a F_2 \sin \theta$$

$$(v) \quad R \tilde{\omega} m = F_2 p_2 \sin \theta + a F_1 - a F_2 \cos \theta$$

$$(vi) \quad R \tilde{\omega} n = 0.$$

Substituting the values of  $Rl$ ,  $Rm$  from (i), (ii) in (iv), (v), we have

$$(F_1 + F_2 \cos \theta) \tilde{\omega} = F_1 p_1 + F_2 p_2 \cos \theta + a F_2 \sin \theta$$

or

$$(vii) \quad F_1 (\tilde{\omega} - p_1) = F_2 (p_2 - \tilde{\omega}) \cos \theta + a F_2 \sin \theta$$

$$F_2 \sin \theta \tilde{\omega} = F_2 p_2 \sin \theta + a F_1 - a F_2 \cos \theta$$

or

$$(viii) \quad a F_1 = F_2 (\tilde{\omega} - p_2) \sin \theta + a F_2 \cos \theta$$

Dividing (vii) by (viii), we have

$$\frac{\tilde{\omega} - p_1}{a} = \frac{(p_2 - \tilde{\omega}) \cos \theta + a \sin \theta}{(\tilde{\omega} - p_2) \sin \theta + a \cos \theta}$$

or

$$(\tilde{\omega} - p_1) (\tilde{\omega} - p_2) \sin \theta + a (\tilde{\omega} - p_1) \cos \theta = a (p_2 - \tilde{\omega}) \cos \theta + a^2 \sin \theta$$

or

$$[a (\tilde{\omega} - p_1 + \tilde{\omega} - p_2)] \cos \theta = [a^2 - (\tilde{\omega} - p_1) (\tilde{\omega} - p_2)] \sin \theta$$

or

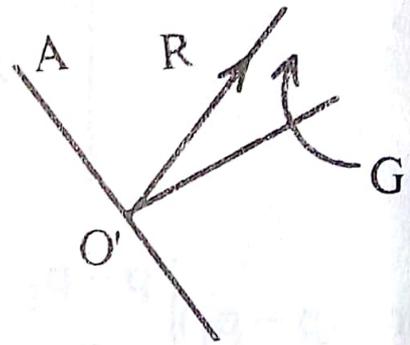
$$\tan \theta = \frac{a (2 \tilde{\omega} - p_1 - p_2)}{a^2 - (\tilde{\omega} - p_1) (\tilde{\omega} - p_2)}$$

or

$$\theta = \tan^{-1} \left[ \frac{a (2 \tilde{\omega} - p_1 - p_2)}{a^2 - (\tilde{\omega} - p_1) (\tilde{\omega} - p_2)} \right].$$

## 9. Nul lines and planes

We know that a system of forces can be reduced to a single force acting through an arbitrarily chosen point  $O'$  and a couple whose axis passes through  $O'$ . Let the resultant force be  $R$  and the resultant couple be  $G$  acting from  $O'$ . Through  $O'$  take any line  $O'A$  perpendicular to the axis of  $G$ . About this line  $O'A$ , the sum of the moments of the forces of the system is zero, for the axis of  $G$  has no component along it and  $R$  meets it. (moment about  $OA = R \cdot 0 + G \cos 90^\circ = 0$ ). Such a line is called **Nul line**, that is, *the line about which the sum of moments of the forces of the system is zero, is called nul line.*



The locus of nul lines is a plane and it is called **nul plane**. Also the

point O' is called the nul point of the plane.

### 10. Equation of nul plane

Let  $X, Y, Z$  be the component forces along  $Ox, Oy, Oz$  and  $L, M, N$  be the component couples about them. Let  $A$  be any point having coordinates  $(\xi, \eta, \zeta)$  referred to the axes  $Ox, Oy, Oz$  and then the coordinates of any point  $P(x_1, y_1, z_1)$  are changed to  $(x_1 - \xi, y_1 - \eta, z_1 - \zeta)$  due to shifting of the axes through  $A$  parallel to the original axes.

Hence, the moment about a line through  $Q$  parallel to  $Ox$  axis is

$$\begin{aligned} L' &= \Sigma \{(y_1 - \eta) Z_1 - (z_1 - \zeta) Y_1\} = \Sigma (y_1 Z_1 - z_1 Y_1) - \eta \Sigma Z_1 + \zeta \Sigma Y_1 \\ &= L - \eta Z + \zeta Y. \end{aligned}$$

Similarly,

$$M' = M - \zeta X + \xi Z, \quad N' = N - \xi Y + \eta X.$$

The quantities  $L', M', N'$  are proportional to the direction cosines of the axis of the resultant couple at  $(\xi, \eta, \zeta)$ . This axis is the normal to the nul plane at  $A$ . The equation of the line through  $A$  are

$$\frac{x - \xi}{l} = \frac{y - \eta}{m} = \frac{z - \zeta}{n}$$

If the line through  $A$  is nul line, it should be perpendicular to the axis of the resultant couple. Thus,

$$(x - \xi)(L - \eta Z + \zeta Y) + (y - \eta)(M - \zeta X + \xi Z) + (z - \zeta)(N - \xi Y + \eta X) = 0$$

or

$$\begin{aligned} x(L - \eta Z + \zeta Y) + y(M - \zeta X + \xi Z) + z(N - \xi Y + \eta X) \\ = \xi(L - \eta Z + \zeta Y) + \eta(M - \zeta X + \xi Z) + \zeta(N - \xi Y + \eta X) \end{aligned}$$

or

$$(10.1) \quad x(L - \eta Z + \zeta Y) + y(M - \zeta X + \xi Z) + z(N - \xi Y + \eta X) = \xi L + \eta M + \zeta N.$$

Thus, the equation (10.1) is the equation to the nul plane of a given point  $(\xi, \eta, \zeta)$  referred to any axes  $Ox, Oy, Oz$ .

### 11. Condition for a straight line to be a nul line (Gkp.'08)

Let the equation of a straight line passing through the point  $A(\xi, \eta, \zeta)$  be

$$\frac{x - \xi}{l} = \frac{y - \eta}{m} = \frac{z - \zeta}{n}$$

The component couples about the lines through A parallel to the axes are  $L - \eta Z + \xi Y$ ,  $M - \zeta X + \xi Z$  and  $N - \xi Y + \eta X$ .

Hence, if the line is nul line the moment of the couple about the given line

$$= l (L - \eta Z + \xi Y) + m (M - \zeta X + \xi Z) + n (N - \xi Y + \eta X)$$

should be zero. Thus,

$$\begin{aligned} lL + mM + nN &= l(-\zeta Y + \eta Z) + m(-\xi Z + \zeta X) + n(-\eta X + \xi Y) \\ &= X(m\zeta - \eta n) + Y(n\xi - l\zeta) + Z(l\eta - m\xi) \end{aligned}$$

or

$$\begin{vmatrix} X & Y & Z \\ l & m & n \\ \xi & \eta & \zeta \end{vmatrix} = Ll + Mm + Nn.$$

This is the condition that the given line may be a nul line of the system.

**Exp. (11.1)** Find the nul point of the plane  $x + y + z = 0$  for the force system  $(X, Y, Z; L, M, N)$ . (Gkp.'06)

**Sol.** In order to find the nul point of the plane

$$(i) \quad x + y + z = 0,$$

we have to compare it with the equation of the nul plane

$$(ii) \quad \begin{aligned} x(L - \eta Z + \zeta Y) + y(M - \zeta X + \xi Z) + z(N - \xi Y + \eta X) \\ = \xi L + \eta M + \zeta N. \end{aligned}$$

So,

$$\frac{L - \eta Z + \zeta Y}{1} = \frac{M - \zeta X + \xi Z}{1} = \frac{N - \xi Y + \eta X}{1} = \frac{\xi L + \eta M + \zeta N}{0}$$

From the ratios first and third, we have

$$L - \eta Z + \zeta Y = N - \xi Y + \eta X$$

or

$$\xi Y - \eta(Z + X) + \zeta Y = N - L$$

or

$$(iii) \quad (\xi + \zeta) Y - \eta(Z + X) = N - L.$$

From the ratios second and third, we have

$$M - \zeta X + \xi Z = N - \xi Y + \eta X$$

or

$$\xi(Y + Z) - \eta X - \zeta X = N - M$$

or

$$(iv) \quad \xi(Y + Z) - (\eta + \zeta)X = N - M.$$

$$(v) \quad \xi L + \eta M + \zeta N = 0.$$

Also, since  $(\xi, \eta, \zeta)$  lies on the plane (i), we have

$$(vi) \quad \xi + \eta + \zeta = 0.$$

On substituting from (vi) in (iii), we have

$$-\eta Y - \eta(Z + X) = N - L$$

or

$$(vii) \quad \eta = \frac{L - N}{(X + Y + Z)}$$

Similarly,

$$(viii) \quad \xi = \frac{N - M}{(X + Y + Z)}.$$

Substituting in (vi) from (vii) and (viii), we have

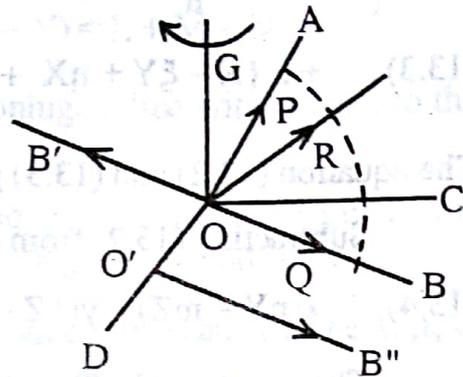
$$\zeta = -(\xi + \eta) = -\frac{L - M}{(X + Y + Z)} = \frac{M - L}{(X + Y + Z)}.$$

Thus, the nul point is

$$\left( \frac{N - M}{X + Y + Z}, \frac{L - N}{X + Y + Z}, \frac{M - L}{X + Y + Z} \right).$$

**12. Replacement of a system of forces by two forces, one of which acts along a given line** (Gkp.'09)

Let  $O$  be the origin, and  $R$  and  $G$  be the resultant force and couple, and  $OA$  is the line along which one of the forces acts. Let a plane  $COD$  be perpendicular to the axis of resultant couple  $G$ , that is the forces of the couple lies in the plane  $COD$ . Now, draw a plane through  $OA$  and the line of action of  $R$ , which cut the plane  $COD$  along the line  $OB$ . Resolve  $R$  into two forces, one  $P$  along  $OA$ , and the other  $Q$  along  $OB$ . Replace the couple  $G$  by two forces  $Q$  along  $OB$  in the opposite direction  $OB'$  and  $O'B''$ , where  $O'$  is such that  $Q \cdot OO' = G$ . Thus, the force  $Q$  along  $OB$  when compounded with the two forces  $Q$ , one along  $OB'$  and the other  $O'B''$  give a force  $Q$  along  $O'B''$ , which is parallel to  $OB$ . As the plane  $COD$  is the nul plane of  $O$ , the sum of the moments of the forces of the system about any line in this plane is zero. Hence, finally the system is reduced to a force  $P$  along  $OA$  and the other force  $Q$  in the nul plane of  $O$  but parallel to  $OB$ . Such forces  $P$  and  $Q$  are called conjugate forces and their lines of action are called conjugate lines.



**13. Equation of a conjugate line**

Suppose we require the conjugate line with respect to the system of the forces given by  $(X, Y, Z; L, M, N)$  of the line

$$(13.1) \quad \frac{x - \xi}{l} = \frac{y - \eta}{m} = \frac{z - \zeta}{n}$$

We know that the nul plane of  $(\xi, \eta, \zeta)$  is (see §10)

$$(13.2) \quad x(L - \eta Z + \zeta Y) + y(M - \zeta X + \xi Z) + z(N - \xi Y + \eta X) = \xi L + \eta M + \zeta N.$$

Another point on (13.1) can be taken as

$$\left( \xi - \frac{\zeta l}{n}, \eta - \frac{\zeta m}{n}, 0 \right)$$

its nul plane is

$$x \left\{ L - \left( \eta - \frac{\zeta m}{n} \right) Z \right\} + y \left\{ M - 0 \cdot X + \left( \xi - \frac{\zeta l}{n} \right) Z \right\} + z \left\{ N - \left( \xi - \frac{\zeta l}{n} \right) Y + \left( \eta - \frac{\zeta m}{n} \right) X \right\} = \left\{ \left( \xi - \frac{\zeta l}{n} \right) L + \left( \eta - \frac{\zeta m}{n} \right) M \right\}.$$

or

$$(13.3) \quad x\left\{L - \eta Z + \frac{\zeta m}{n} Z\right\} + y\left\{M + \xi Z - \frac{\zeta l}{n} Z\right\} + z\left\{N - \xi Y + \eta X + \frac{\zeta l}{n} Y - \frac{\zeta m}{n} X\right\} = \left(\xi - \frac{\zeta l}{n}\right)L + \left(\eta - \frac{\zeta m}{n}\right)M$$

The equation (13.2) and (13.3) give the required conjugate line.

Subtracting (13.2) from (13.3), we have

$$(13.4) \quad x(nY - mZ) + y(lZ - nX) + z(mX - lY) = Ll + Mm + Nn$$

The equation (13.2) and (13.4) give the conjugate line more easily

**Exp. (13.1)** Show that the wrench  $(X, Y, Z; L, M, N)$  is equivalent to two forces, one along the line  $x = y = z$ , and the other along the line given by

$$Lx + My + Nz = 0$$

$$x(Y - Z) + y(Z - X) + z(X - Y) = L + M + N$$

and find the magnitudes of the two forces.

**Sol.** The equation of the given straight line can be written as

$$\frac{x - 0}{1/\sqrt{3}} = \frac{y - 0}{1/\sqrt{3}} = \frac{z - 0}{1/\sqrt{3}}$$

In order to find the conjugate line, we take two points on the given line, and the line of intersection of the nul planes of these points, give the conjugate line.

Let the two points on the given line be  $(0, 0, 0)$  and  $(1, 1, 1)$ . The equation of the nul plane of  $(0, 0, 0)$

$$(i) \quad xL + My + Nz = 0.$$

The equation of nul plane of  $(1, 1, 1)$  is

$$(ii) \quad L'(x - 1) + M'(y - 1) + N'(z - 1) = 0 \quad \begin{matrix} 1 & 1 & 1 \\ X & Y & Z \end{matrix}$$

$$L' = L - (Z - Y) = L - Z + Y$$

$$M' = M - (X - Z) = M - X + Z$$

$$N' = N - (Y - X) = N - Y + X$$

Substituting these values in (ii), we have

$$(L - Z + Y)(x - 1) + (M - X + Z)(y - 1) + (N - Y + X)(z - 1) = 0$$

or  $x(L - Z + Y) + y(M - X + Z) + z(N - Y + X) = L + M + N.$

Subtracting the equation (i) from the above equation, we get

(iii)  $x(Y - Z) + y(Z - X) + z(X - Y) = L + M + N.$

The intersection of (i) and (iii) is conjugate line with respect to the given system of the given line.

Let the two conjugate forces be P and Q. P is along the given line  $x = y = z$  and Q is along second conjugate line given (i) and (iii).

Let the direction cosines of the conjugate line (i) and (iii) be  $\lambda, \mu, \nu.$

Then

(iv)  $L\lambda + M\mu + N\nu = 0$

(v)  $(Y - Z)\lambda + (Z - X)\mu + (X - Y)\nu = 0.$

From these equations, we have

$$\frac{\lambda}{M(X - Y) - N(Z - X)} = \frac{\mu}{N(Y - Z) - L(X - Y)} = \frac{\nu}{L(Z - X) - M(Y - Z)}$$

or

(vi)  $\frac{\lambda}{(M + N)X - (MY + NZ)} = \frac{\mu}{(N + L)Y - (NZ + LX)} = \frac{\nu}{(L + M)X - (LX + MY)}$

The resolved parts along the axes are given by

(vii)  $\frac{P}{\sqrt{3}} + Q\lambda = X, \quad \frac{P}{\sqrt{3}} + Q\mu = Y, \quad \frac{P}{\sqrt{3}} + Q\nu = Z.$

Multiplying these equations by L, M, N respectively and adding, we have

$$\frac{P}{\sqrt{3}}(L + M + N) + Q(\lambda L + \mu M + \nu N) = LX + MY + NZ.$$

or

$$P = \frac{(LX + MY + NZ)\sqrt{3}}{L + M + N} \quad \text{due to (iv).}$$

The other force Q is given by

$$Q^2(\lambda^2 + \mu^2 + \nu^2) = \left(X - \frac{P}{\sqrt{3}}\right)^2 + \left(Y - \frac{P}{\sqrt{3}}\right)^2 + \left(Z - \frac{P}{\sqrt{3}}\right)^2$$