METHOD OF VARIATION OF PARAMETERS

This method is used to find a particular integral whenever the complementary function of the given differential equation is known.

Let the complementary function of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{...(1)}$$

be

$$y = Au + Bv \qquad \text{...(2)}$$

where u and v are functions of x and A and B are constants.

$\because$ u and v are part of C.F.

$\therefore$

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \qquad \text{...(3)}$$

and

$$\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv = 0 \qquad \text{...(4)}$$

Let the complete solution of (1) be

$$y = Au + Bv \qquad \text{...(5)}$$

where A and B are not constants but suitable functions of x to be so choosen that (5) satisfies (1). Now,

$$y_1 = \frac{dy}{dx} = Au_1 + Bv_1 + A_1u + B_1v \qquad \text{...(6)}$$

or

$$y_1 = Au_1 + Bv_1 + (A_1u + B_1v) \qquad \text{...(7)}$$

Let us choose A and B such that

$$A_1u + B_1v = 0 \qquad \text{...(8)}$$

Hence, equation (6) becomes

$$y_1 = Au_1 + Bv_1 \qquad \text{...(9)}$$

$\therefore$

$$y_2 = A_1u_1 + Au_2 + B_1v_1 + Bv_2$$

Substituting the values of y, y_1, y_2 from (5), (8) and (9) in (1) respectively, we get

$$(A_1u_1 + Au_2 + B_1v_1 + Bv_2) + P(Au_1 + Bv_1) + Q(Au + Bv) = R$$

$$A_1u_1 + B_1v_1 + A(u_2 + Pu_2 + Qu) + B(v_2 + Pv_1 + Qv) = R$$

or

Using (3) and (4), we get

$$A_1u_1 + B_1v_1 = R \qquad \qquad \qquad ...(10)$$

Solving (7) and (10) for A_1 and B_1, we get

$$A_1u + B_1v = 0$$

$$A_1u_1 + B_1v_1 - R = 0$$

$$\Rightarrow \qquad \frac{A_1}{-Rv} = \frac{B_1}{Ru} = \frac{1}{uv_1 - u_1v}$$

$$\Rightarrow \qquad A_1 = -\frac{Rv}{uv_1 - u_1v} = \phi(x) \text{ (say)} \qquad ...(11)$$

and

$$B_1 = \frac{Ru}{uv_1 - u_1v} = \psi(x) \text{ (say)} \qquad ...(12)$$

Integrating (11) and (12), we get

$$A = \int \phi(x)\, dx + a \qquad \qquad ...(13)$$

and

$$B = \int \psi(x)\, dx + b \qquad \qquad ...(14)$$

where a and b are arbitrary constants of integration.

Putting the values of A and B in equation (5), we get

$$y = \left[\int \phi(x)\, dx + a\right] u + \left[\int \psi(x)\, dx + b\right] v$$

or

$$y = u \int \phi(x)\, dx + v \int \psi(x)\, dx + au + bv.$$

This gives the complete solution of (1).

Steps for Solution :

1. Write the given equation in the form (if it is not so)

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \qquad ...(1)$$

2. Find the C.F. of equation (1) by taking $R = 0$. Let the C.F. of (1) be

$$y = c_1u + c_1v \qquad \qquad ...(2)$$

where c_1 and c_2 are arbitrary constants and u, v are functions of x.

3. Replacing c_1 and c_2 by A and B in equation (2), where A and B are functions of x. Now consider

$$y = Au + Bv$$

as the complete solution of equation (1).

4. A and B are determined by the formulae

$$A = \int \frac{-Rv}{uv_1 - u_1v}\, dx + a, \quad B = \int \frac{Ru}{uv_1 - u_1v}\, dx + b$$

where a and b are the arbitrary constants of integration.

5. Write $y = Au + Bv$ as the complete solution.

Extension of Variation of Parameter method : The above method can be extended to linear equation of any order.

Consider a differential equation of 3rd order as

$$\frac{d^3y}{dx^3} + P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = S. \qquad \text{...(1)}$$

Let $y = u$, $y = v$ and $y = w$ be the solutions of the given equation when $S = 0$.

Let the complete solution of (1) be

$$y = Au + Bv + Cw$$

where A, B and C are some suitable functions of x.

The conditions to determined A, B, C are

$$A_1u + B_1v + C_1w = 0 \qquad \text{...(2)}$$

$$A_1u_1 + B_1v_1 + C_1w_1 = 0 \qquad \text{...(3)}$$

and $$A_1u_2 + B_1v_2 + C_1w_2 = 0 \qquad \text{...(4)}$$

Solving (2), (3) and (4), we get A_1, B_1 and C_1 which by integration will give A, B and C.

As the solution is obtained by varying the arbitrary constants of the complementary function, the above method is known as that of **variation of parameters.**

Example 1. *Solve by the method of variation of parameters the equation*

$$\frac{d^2y}{dx^2} + a^2y = \sec ax$$

(Gkp. 2011, 13, 15)

Solution. Here, the auxiliary equation is

$$m^2 + a^2 = 0 \quad \text{or} \quad m = \pm ia$$

$$\text{C.F.} = c_1 \cos ax + c_2 \sin ax.$$

Let the complete solution of given differential equation be

$$y = A \cos ax + B \sin ax \qquad \text{...(1)}$$

where A and B are suitable functions of x.

To determine the values of A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1v}\,dx + c_1$$

$$= \int \frac{-\sec ax \cdot \sin ax}{\{\cos ax \cdot a \cos ax - (-a \sin ax) \sin ax\}}\,dx + c_1$$

$$= -\int \frac{\tan ax}{a}\,dx + c_1$$

$$= \frac{1}{a^2} \log \cos ax + c_1.$$

where c_1 is an arbitrary constant of integration.

$$B = \int \frac{Ru}{uv_1 - u_1 v} dx + c_2$$

$$= \int \frac{\sec ax \cdot \cos ax}{\{\cos ax \cdot a \cos ax - (- a \sin ax) \sin ax\}} dx + c_2$$

$$= \frac{1}{a} \int dx + c_2 = \frac{x}{a} + c_2.$$

where c_2 is an arbitrary constant of integration.

Using these results in equation (1), we get

$$y = \left(\frac{1}{a^2} \log \cos ax + c_1 \right) \cos ax + \left(\frac{x}{a} + c_2 \right) \sin ax$$

which is compute solution of the given equation.

$$= \{\sin 2x - \log(\sec 2x + \tan 2x) + c_1\} \cos 2x + (-\cos 2x + c_2) \sin 2x$$

or

$$y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x)$$

where c_1 and c_2 are arbitrary constant of integration.

Example 3. *Solve by the method of variation of parameters :*

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} - 5y = xe^{-x}.$$

(Gkp. 2008, IAS 1991)

given that $y = 0$ *and* $\frac{dy}{dx} = 0$ *when* $x = 0$.

Solution. Here, the auxiliary equation is

$$m^2 - 4m - 5 = 0$$

or

$$(m - 5)(m + 1) = 0 \qquad \Rightarrow \quad m = 5, -1.$$

$$\therefore \qquad \text{C.F.} = c_1 e^{5x} + c_2 e^{-x}$$

i.e., part of C.F. are $u = e^{5x}$, $v = e^{-x}$.

Let

$$y = A e^{5x} + B e^{-x} \qquad \qquad \qquad ...(1)$$

be the complete solution of the given equation where A and B are determined as :

$$A = \int \frac{-Rv}{uv_1 - u_1 v} \, dx + c_1$$

$$= \int \frac{-xe^{-x} \cdot e^{-x}}{-6 e^{4x}} \, dx + c_1 \qquad (\because R = xe^{-x})$$

$$= \frac{1}{6} \int x e^{-6x} \, dx + c_1$$

$$= -\frac{1}{36}(6x + 1) e^{-6x} + c_1.$$

and

$$B = \int \frac{Ru}{uv_1 - u_1 v} \, dx + c_2.$$

$$= \int \frac{xe^{-x} \cdot e^{5x}}{-6 e^{4x}} \, dx + c^2$$

$$= -\frac{1}{6}\left(\frac{x^2}{2}\right) + c_2 = -\frac{1}{12} x^2 + c_2.$$

Hence, from (1) the complete solution is

$$y = \left\{-\frac{1}{36}(6x + 1) e^{-6x} + c_1\right\} e^{5x} + \left(-\frac{1}{12} x^2 + c_2\right) e^{-x}$$

or

$$y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{36}(6x + 1) e^{-x} - \frac{1}{12} x^2 e^{-x}.$$

or

$$y = c_1 e^{5x} + c_2 e^{-x} - \frac{1}{36}(x^2 + 6x + 1) e^{-x} \qquad ...(2)$$

Differentiating equation (2) with respect to x, we get

Example 4. *Solve by the method of variation of parameters :*

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = x^2 e^x.$$

Solution. The given equation is

$$x^2 y'' + xy' - y = x^2 e^x \qquad \qquad \text{...(1}$$

or

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = e^x \qquad \qquad \text{...(2}$$

Here, $R = e^x$.

Consider the equation $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$ for finding parts of C.F.

Put $x = e^z$ so that $z = \log x$. Then the above equation reduces to

$$[D(D-1) + D - 1] y = 0 \quad \text{where } D \equiv \frac{d}{dz}.$$

or

$$(D^2 - 1) \, y = 0$$

Auxiliary equation is

$$m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$$

$$\therefore \quad \text{C.F.} = c_1 e^z + c_2 e^{-z} = c_1 x + c_2 \cdot \frac{1}{x}.$$

Hence part of C.F. are x and $\frac{1}{x}$.

Let $u = x$ and $v = 1/x$.

Let $y = Ax + \frac{B}{x}$ be the complete solution, where A and B are function of x.

A and B are determined as :

$$A = \int \frac{-Rv}{uv_1 - u_1 v} \, dx + c_1 = -\int \frac{e^x \cdot \frac{1}{x}}{x\left(-\frac{1}{x^2}\right) - 1 \cdot \left(\frac{1}{x}\right)} \, dx + c_1$$

$$= \frac{1}{2} \int e^x \, dx + c_1 = \frac{1}{2} e^x + c_1.$$

and $\quad B = \int \frac{Ru}{uv_1 - u_1 v} \, dx + c_2 = \int \frac{e^x \cdot x}{x\left(-\frac{1}{x^2}\right) - 1 \cdot \left(\frac{1}{x}\right)} \, dx + c_2$

$$= -\frac{1}{2} \int x^2 e^x \, dx + c_2$$

$$= -\frac{1}{2} x^2 e^x + (x - 1) e^x + c_2.$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2} e^x + c_1\right) x + \left[-\frac{1}{2} x^2 e^x + (x - 1) e^x + c_2\right] \cdot \frac{1}{x}$$

or $\qquad y = c_1 x + \frac{c_2}{x} + \left(1 - \frac{1}{x}\right) e^x.$

where c_1 and c_2 are arbitrary constants of integration.