4

Motion in A Resisting Medium

(Straight line only)

§ 4.1. INTRODUCTION

Every body moving in a medium like air feels a resistance to its motion and this resistance increases as the velocity of the body increases and so it (resistance) may be assumed to be equal to some function of the velocity of the body.

So far the actual law of this resistance could not be discovered but for projectiles moving with velocities under 800 ft/sec. the resistance is taken to vary as the square of the velocity whereas for projectiles moving with velocities between 800 ft./sec. and 1350 ft./sec. the resistance is taken to vary as the cube or even a higher power of the velocity. But as the velocity of the projectile increases beyond 1350 ft./sec then the resistance again appears to be varying as the square of the velocity.

This resisting force always acts opposite to the direction of motion and is non-conservative and so the principle of conservation of energy is not applicable.

Terminal Velocity : If a particle is falling under gravity in a resisting medium, then the velocity V, when the downward acceleration is zero, is called the 'terminal velocity'. It is also known as limiting velocity. *(Gorakhpur 2010, 12, 15)*

§ 4.2 MOTION OF A PARTICLE FALLING FREELY UNDER GRAVITY

A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to find the motion.

(Gorakhpur 2005, 07, 09, 11, 12)

Let the particle of mass m (say) be at a distance x from the starting point after time t. Let v be its velocity at this position. The force due to gravity acting vertically downwards on the particle is mg. The resistance varies as the square of the velocity and therefore the force of resistance on the particle is mkv^2 acting against the direction of motion and hence acting vertically upwards.

$\therefore$ The equation of motion of the particle is

$$m\ddot{x} = mg - mkv^2$$

or

$$\ddot{x} = g - kv^2 = g\left(1 - \frac{k}{g}v^2\right) \quad \text{...(i)}$$

Let V be the terminal velocity of the particle *i.e.*, Let V be the velocity of the particle when its acceleration is zero in the vertically downwards motion.

Then from (i) we get $0 = g\left(1 - \frac{k}{g}V^2\right)$ or $\frac{k}{g} = \frac{1}{V^2}$ ...(ii)

$\therefore$ From (i) the equation of motion reduces to

$$\frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{V^2}\right) \qquad \text{...(iii)}$$

or $$v\frac{dv}{dx} = g\left(1 - \frac{v^2}{V^2}\right) = \frac{g}{V^2}(V^2 - v^2) \text{ or } \frac{-2v\,dv}{V^2 - v^2} = \frac{-2g}{V^2}dx$$

Integrating we get $\log(V^2 - v^2) = -\frac{2g}{V^2}x + c$,

where c is constant of integration.

Initially $x = 0, v = 0, \therefore \log V^2 = c$

$\therefore$ From (iv) we get $\log(V^2 - v^2) = -\frac{2g}{V^2}x + \log V^2$

or $$\log(V^2 - v^2) - \log V^2 = -\frac{2g}{V^2}x \text{ or } \log\left(\frac{V^2 - v^2}{V^2}\right) = -\frac{2g}{V^2}x$$

or $$\left(\frac{V^2 - v^2}{V^2}\right) = e^{-2gx/V^2} \text{ or } 1 - \frac{v^2}{V^2} = e^{-2gx/V^2}$$

or $$v^2 = V^2(1 - e^{-2gx/V^2}) \qquad \text{...(v)}$$

which gives the velocity of the particle in any position.

Again from (iii) we have $\frac{dv}{dt} = g\left(1 - \frac{v^2}{V^2}\right)$, $\because \frac{d^2x}{dt^2} = \frac{dv}{dt}$

or $$\frac{dv}{dt} = g\left(\frac{V^2 - v^2}{V^2}\right) \text{ or } \frac{g}{V^2}dt = \frac{dv}{V^2 - v^2}$$

Integrating we get $\frac{g}{V^2}t = \frac{1}{2V}\log\frac{V+v}{V-v} + C_1$, ...(vi)

where C_1 is constant of integration,

Intially, $t = 0, v = 0, \therefore C_1 = 0$

$\therefore$ From (vi) we get $\frac{g}{V^2}t = \frac{1}{2V}\log\left(\frac{V+v}{V-v}\right) = \frac{1}{V}\tanh^{-1}\left(\frac{v}{V}\right)$

or $$\frac{v}{V} = \tanh\left(\frac{g}{V}t\right) \text{ or } v = V\tanh\left(\frac{g}{h}t\right), \qquad \text{...(vii)}$$

which gives velocity of the particle at any time. ***(Gorakhpur 2006)***

Eliminating v from (v) and (vii) we get

$$V^2(1 - e^{-2gx/V^2}) = V^2\tanh^2(gt/V)$$

or $$1 - e^{-2gx/V^2} = \tanh^2(gt/V)$$

or $$e^{-2gx/V^2} = 1 - \tanh^2(gt/V)$$

or $$e^{-2gx/V^2} = \text{sech}^2(gt/V)$$

or $$e^{-2gx/V^2} = \cosh^2(gt/V)$$

$$\frac{2gx}{V^2} = \log \cosh^2 \left(\frac{gt}{V}\right)$$

or

$$\frac{2gx}{V^2} = 2 \log \cosh \left(\frac{gt}{V}\right)$$

or

$$x = \left(\frac{V^2}{g}\right) \log \cosh (gt/V),$$

or

which gives the position of the particle at any time.

§ 4.3. MOTION OF A PARTICLE PROJECTED UPWARDS

A particle is projected vertically upwards under gravity, supposed constant in a resisting medium whose resistance varies as the square of the velocity; to find the motion. *(Gorakhpur 2008, 10, 12)*

Let the particle of mass m (say) be at a distance x from the starting point after time t. Let v be the velocity at this position. The particle is moving upwards and its weight mg is acting vertically downwards. The resistance varies as the square of the velocity and therefore the force of resistance on the particle is mkv^2 acting against the direction of motion and hence acting vertically downwards.

The equation of motion of the particle is

$$m \frac{d^2x}{dt^2} = -mg - mkv^2$$

or

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{k}{g}v^2\right) \quad \text{...(i)}$$

Let V be the terminal velocity of the particle, we put $V^2 = g/k$.

$\therefore$ From (i) we get $$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right) = -\frac{g}{V^2}(V^2 + v^2) \quad \text{...(ii)}$$

Now (ii) can be written as

$$\frac{vdv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$

or

$$\frac{2vdv}{V^2 + v^2} = -\frac{2g}{V^2}dx,$$

Integrating, $$\log (V^2 + v^2) = -\frac{2g}{V^2}x + C, \quad \text{...(iii)}$$

where C is constant of integration.

Initially $x = 0$ and let $v = u$, then $C = \log (V^2 + u^2)$

$\therefore$ From (iii) we have $\log (V^2 + v^2) = -(2g/V^2)x + \log (V^2 + u^2)$

or

$$\frac{2g}{V^2}x = \log\left(\frac{V^2 + u^2}{V^2 + v^2}\right). \quad \text{...(iv)}$$

which gives velocity of the particle at any position.

Again from (ii) we get $\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2)$, since $\frac{d^2x}{dt^2} = \frac{dv}{dt}$

or $$-\frac{g}{V^2}dt = \frac{dv}{V^2 + v^2}$$

Integrating, $$-\frac{g}{V^2}t = \frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) + C_1 \quad ...(v)$$

where C_1 is constant of integration.

Initially $t = 0, v = u$, $\therefore \quad C_1 = -\frac{1}{V}\tan^{-1}\left(\frac{u}{V}\right)$

$\therefore$ From (v) we get $$-\frac{gt}{V^2} = \frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) - \frac{1}{V}\tan^{-1}\left(\frac{u}{V}\right)$$

or $$t = \frac{V}{g}\left[\tan^{-1}\left(\frac{u}{V}\right) - \tan^{-1}\left(\frac{v}{V}\right)\right], \quad ...(vi)$$

which is a relation between time and velocity.

§ 4.4 VERTICAL MOTION WHEN RESISTANCE VARIES AS THE VELOCITY

(A) A particle falls under gravity (supposed constant) in a resisting medium when resistance varies as the velocity; to find the motion if the particle starts from rest.

Let v be the velocity when the particle has fallen a distance x in time t. The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - mkv \quad ...(1)$$

where mkv is the resistance and m the mass since $v = (g/k)$ makes $\frac{d^2x}{dt^2} = 0$,

$(g/k) = V$ (say) is the terminal velocity. Hence from (1)

$$v\frac{dv}{dx} = \frac{dv}{dt} = g\left(1 - \frac{v}{V}\right) \quad ...(2)$$

or $$\frac{v\,dv}{V - v} = \frac{g}{V}dx$$

Integrating, $$-v - V\log(V - v) = \frac{g}{V}x + A.$$

Initially $x = 0$, $v = 0$, therefore, $A = -V\log V$

Hence, $$-v - V\log\left(\frac{V - v}{V}\right) = \frac{g}{V}x.$$

or $$gx = -vV - V^2\log\left(1 - \frac{v}{V}\right) \quad ...(3)$$

This is the relation between x and v.

Again from (2),

$$\frac{dv}{V - v} = \frac{g}{V}dt$$

which on integration, gives

$$\log(V - v) = -\frac{g}{V}t + B.$$

Initially $t = 0$, $v = 0$, therefore, $B = \log V$

Hence $\log\left(\frac{V-v}{V}\right) = -\frac{g}{V}t$

or $$v = V(1 - e^{-gt/V}) \quad ...(4)$$

This result shown that as $t \to \infty$, v increases and tends to V the terminal velocity.

From (4) $$\frac{dx}{dt} = V(1 - e^{-gt/V})$$

which on integration, gives

$$x = Vt + \frac{V^2}{g}e^{-gt/V} + C$$

Initially $t = 0, x = 0$, therefore $C = -\frac{V^2}{g}$.

Hence, $$x = Vt - \frac{V^2}{g}(1 - e^{-gt/V}) \quad ...(5)$$

Equation (4) and (5) give velocity and distance fallen by the particle at the end of any time t.

(B) A particle is projected upwards under gravity (supposed constant) in a resisting medium whose resistance varies as the velocity; to find the motion.

Let U be the velocity of projection, v the velocity at time t when the particle has risen up a distance x above the point of projection. The equation of motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv$$

or $$v\frac{dv}{dx} = \frac{dv}{dt} = -g\left(1 + \frac{v}{V}\right) \quad ...(1)$$

where $V = (g/k)$ is the terminal velocity'

Therefore,

$$\frac{v\,dv}{V + v} = -\frac{g}{V}dx,$$

which on integration gives

$$v - V\log(V + v) = -\frac{g}{V}x + D.$$

Initially $x = 0$, $v = U$, therefore

$$D = U - V\log(V + U)$$

Hence, $$v = \frac{V^2}{g}\log\frac{V+v}{V+U} + \frac{V}{g}(U - v) \quad ...(2)$$

At the maximum height $v = 0$, therefore the greatest height

$$h = \frac{VU}{g} + \frac{V^2}{g}\log\frac{V}{V+U} \quad ...(3)$$

Again, from (1), $\frac{dv}{V + v} = -\frac{g}{V}dt$

which on integration, gives

$$\log (V + v) = -\frac{g}{V}t + E.$$

Initially, $t = 0, v = U$, therefore $E = \log (U + V)$

Hence, $\log \frac{V+v}{V+U} = -\frac{g}{V}t$

or $$v = (V + U)\, e^{-gt/V} - V \qquad ...(4)$$

or $$\frac{dx}{dt} = (V + U)\, e^{-gt/V} - V$$

Integrating, we get

$$x = -\frac{V}{g}(V + U)\, e^{-gt/V} - Vt + F$$

Initially $t = 0$, $x = 0$, therefore $F = \frac{V}{g}(V + U)$

Hence,

$$x = \frac{V}{g}(V + U)\,(1 - e^{-gt/V}) - Vt \qquad ...(5)$$

Equation (2) gives the velocity at height x above the point of projections.

Equation (4) and (5) give the velocity and the height risen at the end of any time t.

ILLUSTRATIVE EXAMPLES

Example 1. A particle is projected vertically upwards with a velocity *u* in a medium, the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.

Solution. Let the particle of mass m (say) be at a distance x from starting point after time t. Let v be its velocity at this position. The particle is moving upwards and its weight mg is acting vertically downwards. The resistance varies as the cube of the velocity and therefore the force of resistance on the particle is mkv^3, acting against the direction of motion and hence acting vertically downwards.

The equation of motion of the particle is

$$m\frac{c^2x}{dt^2} = -mg - mkv^3 \quad \text{or} \quad \frac{vdv}{dx} = -g\left(1 + \frac{k}{g}v^2\right), \qquad ...(i)$$

since $d^2x/dt^2 = v\,(dv/dx)$.

Let V be the terminal velocity of the particle. If the particle be moving vertically downwards in this medium (g9ven), its equation of motion is

$$m\left(\frac{d^2x}{dt^2}\right) = mg - mkv^3$$

$\therefore\ 0 = mg - mkV^3$, where V is its terminal velocity.

or $$g = kV^3 \quad \text{or} \quad g/k = V^3 \qquad ...(ii)$$

$\therefore$ From (i) we get $v\, dv/dx) = -g[1 + (v^3/V^3)]$, put $\frac{g}{k} = V^3$

or
$$\frac{v\,dv}{v^3 + V^3} = -\frac{g}{V^3}\,dx \qquad \text{...(iii)}$$

Now
$$\frac{v}{v^3 + V^3} = \frac{v}{(v+V)(v^2 - vV + V^2)}$$

$$\equiv \frac{A}{(v+V)} + \frac{Bv + C}{v^2 - vV + V^2} \qquad \text{...(iv)}$$

or
$$v = A(v^2 - vV + V^2) + (Bv + C)(v + V)$$
$$v = (A + B)v^2 + (-AV + BV + C)v + (AV^2 + CV)$$

Equating the coefficients v^2 and v and constant terms on both sides, we get $0 = A + B;\ 1 = -AV + BV + C;\ 0 = AV^2 + CV.$

Solving these we get $A = -(1/3V) = -B$ and $C = \frac{1}{3}$

$\therefore$ From (iv) we get
$$\frac{v}{(v+V)(v^2 - vV + V^2)} = -\frac{1}{3V(v+V)} + \frac{(1/3V)v + 1/3}{v^2 - vV + V^2}$$
$$= -\frac{1}{3V}\left[\frac{1}{v+V} - \frac{v+V}{v^2 - vV + V^2}\right]$$

$\therefore$ From (iii) we get
$$-\frac{g}{V^3}\,dx = -\frac{1}{3V}\left[\frac{1}{v+V} - \frac{v+V}{v^2 - vV + V^2}\right]dv$$

or
$$\frac{3g}{V^2}\,dx = \frac{1}{v+V}\,dv - \frac{1}{2}\,\frac{(2v + 2V)\,dV}{v^2 - vV + V^2}$$

or
$$\frac{3g}{V^2}\,dx = \frac{dv}{v+V} - \frac{1}{2}\left\{\frac{(2v - V) + 3V}{v^2 - vV + V^2}\right\}dv$$

Integrating we have
$$\frac{3g}{V^2} + C = \log(v + V) - \frac{1}{2}\int\frac{(2v - V)\,dv}{v^2 - vV + V} - \frac{3}{2}V\int\frac{dv}{v^2 - vV + V}$$

$$= \log(v + V) - \frac{1}{2}\log(v^2 - vV + V^2) - \frac{3}{2}V\int\frac{d}{(v - \frac{1}{2}V)^2 + \frac{3}{4}V^2}$$

$$= \log(v + V) - \frac{1}{2}\log(v^2 - vV + V^2) - \frac{3}{2}V\frac{1}{(\frac{1}{2}V\sqrt{3})}\tan^{-1}\left(\frac{v - \frac{1}{2}V}{\frac{1}{2}V\sqrt{3}}\right)$$

or
$$\frac{3gx}{V^2} + C = \log(v + V) - \frac{1}{2}\log(v^2 - vV + V^2) - \sqrt{3}\tan^{-1}\left(\frac{2v - V}{V\sqrt{3}}\right) \qquad \text{...(v)}$$

Initially $x = 0, y = u$ (given)

$$\therefore\ C = \log(u + V) - \frac{1}{2}\log(u^2 - uV + V^2) - \sqrt{3}\tan^{-1}\left(\frac{2u - V}{V\sqrt{3}}\right) \qquad \text{...(vi)}$$

Subtracting (vi) from (v), we get

$$\frac{3gx}{V^2} = \log\left(\frac{v+V}{u+V}\right) - \frac{1}{2}\log\left(\frac{v^2 - vV + V^2}{u^2 - uV + V^2}\right)$$
$$+ \sqrt{3}\left[\tan^{-1}\left(\frac{2u-V}{V\sqrt{3}}\right) \tan^{-1}\left(\frac{2v-V}{V\sqrt{3}}\right)\right]$$

If h be the required height, then at $x = h, v = 0$ and we have

$$\frac{3gh}{V^2} = \log\left(\frac{V}{u+V}\right) - \frac{1}{2}\log\left(\frac{V^2}{u^2 - uV + V^2}\right)$$
$$+ \sqrt{3}\left[\tan^{-1}\left(\frac{2u-V}{V\sqrt{3}}\right) + \tan^{-1}\left(\frac{2v-V}{V\sqrt{3}}\right)\right]$$

or
$$h = \frac{V^2}{3g}\left[\log\left\{\left(\frac{V}{u+V}\right)\cdot\left(\frac{u^2 - uV + V^2}{V^2}\right)^{1/2}\right\}\right.$$
$$\left. + \sqrt{3}\tan^{-1}\left\{\frac{\left(\frac{2u-V}{V\sqrt{3}}\right) + \left(\frac{V}{V\sqrt{3}}\right)}{1 - \left(\frac{2u-V}{V\sqrt{3}}\right)\left(\frac{V}{V\sqrt{3}}\right)}\right\}\right]$$
$$= \frac{V^2}{g}\left[\log\left\{\frac{(u^2 - uV + V^2)^{1/2}}{u+V}\right\} + \sqrt{3}\tan^{-1}\left\{\frac{u\sqrt{3}}{2V - u}\right\}\right].$$

Example 2. A particle of unit mass is projected verticaly upwards with velocity V in a medium for which the resistance is kv when the speed of the particle is v. Prove that the particle returns to the point of projection with speed V_1 such that

$$V + V_1 = \frac{g}{k}\log\left(\frac{g + kV}{g - kV_1}\right)$$

Solution. Let the particle be projected from a point O and let it be at P at a distance x from O at any instant. Let v be the velocity of the particle at P.

The forces acting on the particle at P are the weight $1.g$ and the force due to resistance kv both acting vertically downwards *i.e.* in the sense in which x decreases, therefore the equation of motion of the particle is $1.v\ (dv/dx) = -1.g - kv$.

or
$$\left(\frac{v}{g + kv}\right) dv = -\,dx \quad \text{or} \quad \left(\frac{g + kv - g}{g + kv}\right) dv = -\,k.dx$$

or
$$\left(1 - \frac{g}{g + kv}\right) dv = -\,k.dx$$

Integrating, $\quad v - (g/k)\log(g + kv) = -kx + C \qquad$ ...(i)

Initially $\quad v = V, x = 0 \qquad \therefore\ C = V - (g/k)\log(g + kV)$

$\therefore$ From (i) we get

$$v = (g/k)\log(g + kv) = -kx + V - (g/k)\log(g + KV) \qquad \text{...(ii)}$$

When the particle reaches the highest point of its path, $v = 0$ and $x = h$ (say). Then from (i) we get

$$-kh + V - (g/k)\log(g + kV) = -(g/k)\log(g) \quad \text{...(iii)}$$

Now when the particle returns from the highest point it starts with zero velocity and let v be its velocity when it has fallen a distance y from the highest point. Then the equation of motion of the particle is $1.v\,(dv/dy) = 1.g - kv$

or $$\left(\frac{v}{g - kv}\right) dv = dy$$

or $$\left(\frac{g - kv - g}{g - kv}\right) dv = -k\,dy$$

or $$\left(1 - \frac{g}{g - kv}\right) dv = -k\,dy$$

Integrating, $v + (g/k)\log(g - kv) = -ky + C_1$

Initially, $v = 0,\ y = 0.$ $\quad \therefore\ C_1 = (g/k)\log(g)$

$$\therefore\quad v + (g/k)\log(g - kv) = -ky + (g/k)\log(g) \quad \text{...(iv)}$$

It is given that the particle reaches the point of projection O with a velocity V_1 *i.e.*, at $O, v = V_1$ and $y = h$, so from (iv) we have

$$V_1 + (g/k)\log(g - kV_1) = -kh + (g/k)\log(g) \quad \text{...(v)}$$

Adding (iii) and (v) we get

$$[-kh + V - (g/k)\log(g + kV)] + [V_1 + (g/k)\log(g - kV_1)] = -kh$$

or $$V + V_1 = (g/k)[\log(g + kV) - \log(g - kV_1)]$$

or $$V + V_1 = \frac{g}{k}\log\left(\frac{g + kV}{g - kV_1}\right)$$

Example 3. A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{gm^2}{\mu^2}\left[\frac{\mu t}{m} + e^{-\mu t/m} - 1\right]$$

Solution. The resistance at any instant to the motion is μv, where v is the velocity of the particle at that intant. The resistance is acting verticaly upwards as the particle is falling under gravity. The weight of the particle mg is acting vertically downwards.

$\therefore$ The equation of motion is

$$m\frac{d^2x}{dt^2} = mg - \mu v \quad \text{or} \quad \frac{d^2x}{dt^2} = g - \frac{\mu}{m}v \quad \text{or} \quad \frac{dv}{dt} = g - \frac{\mu}{m}v, \ \because\ \frac{d^2x}{dt^2} = \frac{dy}{dt}$$

or $$\frac{dv}{dt} + \frac{\mu}{m}v = g,$$

which is a linear equation in v and whose integrating factor is

$$e^{\int (\mu/m)\,dt} = e^{\mu t/m}.$$

$\therefore$ Its solution is $ve^{\mu t/m} = c + \int ge^{\mu t/m}\,dt$

or $$v = ce^{-\mu t/m} + ge^{-\mu t/m}\left[\frac{m}{\mu}e^{\mu t/m}\right]$$

or $$v = ce^{-\mu t/m} + (gm/\mu) \quad ...(i)$$

Initially $t = 0,\ v = 0,\ \therefore\ c = -(gm/\mu)$

$\therefore$ From (i) we get, $$v = \frac{gm}{\mu}[1 - e^{-\mu t/m}]$$

or $$\frac{dx}{dt} = \frac{gm}{\mu}[1 - e^{-\mu t/m}]$$

or $$dx = \frac{gm}{\mu}[1 - e^{-\mu t/m}]\,dt$$

Integrating, $x = \frac{gm}{\mu}\left[1 + \frac{m}{\mu}c^{-\mu t/m}\right] + C_1$, where C_1 is constant of integration.

Initially $x = 0,\ t = 0\ \therefore\ C_1 = -gm^2/\mu^2$

Hence $$x = \frac{gm}{\mu}\left[1 + \frac{m}{\mu}e^{-\mu t/m}\right] - \frac{gm^2}{\mu^2}$$

or $$x = \frac{gm}{\mu}\left[1 + \frac{m}{\mu}e^{-\mu t/m} - \frac{m}{\mu}\right]$$

$$= \frac{gm^2}{\mu^2}\left[\frac{\mu t}{m} + e^{-\mu t/m} - 1\right].$$

Example 4. A particle of mass m is projected vertically under gravity the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $(V^2/g)\ [\lambda - \log(1 + \lambda)]$, where V is the terminal velocity of the particle and λV is the initial velocity.

(Gorakhpur 2007, 12, 16)

Solution. The particle is projected upwards so the force of resistance on the particle is acting vertically downwards and therefore the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -mg - mkv \quad \text{or} \quad v\frac{dv}{dx} = -g - kv,\ \because\ v\frac{dv}{dx} = \frac{d^2x}{dt^2} \quad ...(i)$$

Now if the particle is falling vertically downwards, then the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - mkv. \quad ...(ii)$$

Now it is given that the terminal velocity is V *i.e.* the velocity of the particle is V when its acceleration is zero in the downwards motion under gravity. Therefore from (ii) we get $mg - mkV = 0$

or $$V = g/k$$

We put $v = g/k$ in (i) and then

$\therefore$ From (i) we get

$$v\frac{dv}{dx} = -g\left(1 + \frac{k}{g}v\right) = -g\left(1 + \frac{v}{V}\right)$$

or
$$\frac{v\,dv}{v+V} = -\frac{g}{V}dx$$

or
$$-\frac{g}{V}dx = \left(1 - \frac{V}{v+V}\right)dv$$

Integrating we have $-(g/V)x = v - V\log(v+V) + c$, ...(iv)

where c is constant of integration.

Initially $x = 0$ and $v = \lambda V$, $\therefore$ $0 = \lambda V - V\log(\lambda V + V) + c$

or
$$c = V\log(\lambda V + V) - \lambda V$$

$\therefore$ From (iv) we get

$$-(g/V)x = v - V\log(v+V) + V\log(\lambda V + V) - \lambda V \quad ...(v)$$

Let y be the greatest height attained by the particle, then at

$$x = y, \quad v = 0$$

$\therefore$ From (v) we get

$$-(g/V)y = -V\log V + V\log(\lambda V + V) - \lambda V$$
$$= V[\log(\lambda V + V) - \log V] - \lambda V$$
$$= V\log\left(\frac{\lambda V + V}{V}\right) - \lambda V = V\log(\lambda + 1) - \lambda V$$

or
$$y = (V^2/g)[\lambda - \log(1+\lambda)].$$

Example 5. A particle projected upwards with a velocity U, in a medium whose resistance varies as the square of the velocity; will return to the point of projection with velocity $v_1 = UV/\sqrt{(U^2 + V^2)}$ after a time $\frac{V}{g}\left(\tan^{-1}\frac{u}{V} + \tanh^{-1}\frac{v_1}{V}\right)$, where V is the terminal velocity.

Solution. If the particle be coming downwards, then the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - mkv^2, \text{ since resistance is given as } kv^2.$$

$\therefore$ If V be the terminal velocity, then we have $0 = mg - mkV^2$

or
$$V^2 = g/k \qquad \text{or} \qquad k = g/V^2 \quad ...(i)$$

Now if the particle be moving upwards, then the equation of motion is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2 \quad ...(ii)$$

or
$$v\frac{dv}{dx} = -(g + kv^2), \qquad \because \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

or
$$\frac{v\,dv}{g + kv^2} = -dx \qquad \text{or} \qquad \frac{v\,dv}{g + (gv^2/V^2)} = -dx, \text{ from (i)}$$

or
$$\frac{2v\,dv}{V^2 + v^2} = -\frac{2g}{V^2}dx$$

Integrating we have $\log(V^2 + v^2) = -(2g/V^2)x + C$, ...(iii)

where C is constant of integration.

Initially $x = 0$, $v = u$ (given) $\because$ $C = \log(V^2 + u^2)$

$\therefore$ From (iii) we get

$$\log(V^2 + v^2) = -(2g/V^2)x + \log(V^2 + u^2)$$

or $$(2g/V^2)x = \log\left(\frac{V^2 + u^2}{V^2 + v^2}\right) \quad ...(v)$$

Again from (ii) we have

$$\frac{dv}{dt} = -(g + kv^2)$$

or $$\frac{dv}{dt} = -\left(g + \frac{g}{V^2}v^2\right), \text{ from (i)}$$

or $$\frac{dv}{V^2 + v^2} = -\frac{g}{V^2}dt$$

Integrating we have $\frac{1}{V}\tan^{-1}\frac{v}{V} = -\frac{g}{V^2}t + C_1$

where C_1 is constant of integration.

Initially $v = u,\ t = 0,\ C_1 = \frac{1}{V}\tan^{-1}\frac{u}{V}$

$\therefore$ From (v) we get $\frac{1}{V}\tan^{-1}\left(\frac{v}{V}\right) = -\frac{g}{V^2}t + \frac{1}{V}\tan^{-1}\left(\frac{u}{V}\right)$

or $$t = \frac{V}{g}\left[\tan^{-1}\frac{u}{U} - \tan^{-1}\frac{v}{V}\right] \quad ...(vi)$$

Let t_1 be the time taken by the particle in reaching the highest point of its path where its velocity is zero.

Then from (vi) we get $t_1 = \frac{V}{g}\left[\tan^{-1}\frac{u}{V}\right]$ ...(vii)

Also let h be the maximum height attained by the particle *i.e.*, $x = h$ when $v = 0$. Then from (iv) we have

$$\frac{2gh}{V^2} = \log\left(\frac{V^2 + u^2}{V^2}\right) \text{ or } h = \frac{V^2}{2g}\log\left(\frac{V^2 + u^2}{V^2}\right) \quad ...(viii)$$

From the highest point the particle will start moving downwards from rest under gravity in the resisting medium and therefore the equation of motion of particle while falling downwards is

$$mv\frac{dv}{dx} = mg - mkv^2 = mg - m\left(\frac{g}{V^2}\right)v^2, \text{ from (i)}$$

or $$v\frac{dv}{dx} = g\left(\frac{V^2 - v^2}{V^2}\right) \quad ...(ix)$$

or $$\frac{2v\,dv}{V^2 - v^2} = \frac{2g}{V^2}dx \quad ...(x)$$

Integrating we have $-\log(V^2 - v^2) = (2g/V^2)x + C_2$,

where C_2 is constant of integration.

Initially (*i.e.* at the highest point) $x = 0, v = 0$. $\therefore$ $C_2 = -\log V^2$

$\therefore$ From (x) we get

$$(2g/V^2)\,x = \log V^2 - \log (V^2 - v^2) \qquad \text{...(xi)}$$

Let v_1 be the velocity of the particle when it reaches back the point of projection then at $x = h, v = v_1$ and so from (xi) we get

$$\frac{2g}{V^2}h = \log\left(\frac{V^2}{V^2 - v_1^2}\right)$$

or $$\log\left(\frac{V^2 + u^2}{V^2}\right) = \log\left(\frac{V^2}{V^2 - v_1^2}\right), \text{ from (viii)}$$

or $$\frac{V^2 + u^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$$

or $$(V^2 + u^2)(V^2 + v_1^2) = V^4$$

or $$u^2V^2 - v_1^2 (V^2 + u^2) = 0$$

or $$v_1^2 (V^2 + u^2) = u^2V^2 \qquad \text{...(A)}$$

or $$v_1 = uV/\sqrt{(V^2 + u^2)}$$

Again from (ix) we have

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2) \quad \text{or} \quad \frac{g}{V^2}dt = \frac{dv}{V^2 - v^2}$$

Integrating we hve $\quad \dfrac{g}{V^2}t = \dfrac{1}{V}\tanh^{-1}\dfrac{v}{V} + C_3, \qquad$ ...(xii)

when C_3 is constant of integration.

Initially (i.e. at the highest point) $t = 0,\ v = 0, \quad \therefore\ C_3 = 0$

$\therefore$ From (xii) we get $\quad t = (V/g)\tanh^{-1}(v/V) \qquad$ ...(xiii)

$\therefore$ When the particle reaches the point of projection, $v = v_1$ and $t = t_2$ say.

Then from (xiii) we get $\quad t_2 = (V/g)\tanh^{-1}(v_1/V) \qquad$ ...(xiv)

Hence from (vii) and (xiv), the required time

$$= t_1 + t_2 = \frac{V}{g}\left[\tan^{-1}\frac{u}{V} + \tanh^{-1}\frac{v_1}{V}\right].$$

Example 6. A particle moving in a straight line is subject to a resistance kv^3, where v is the velocity. Show that if v is the velocity at time t when he distance is s, $v = u/(1 + kus)$; $t = (s/u) + \frac{1}{2}ks^2$, where u is the initial velocity.

(Gorakhpur 2014)

Solution. The equation of motion of the particle is

$$\frac{d^2s}{dt^2} = -kv^3 \quad \text{or} \quad v\frac{dv}{dx} = -kv^3, \qquad \because \frac{d^2s}{dt^2} = v\frac{dv}{ds}$$

or $$(-1/v^2)\,dv = k\,ds$$

Integrating we have $\ 1/v = ks + c$, where c is constant of integration.

Initially $s = 0$ and $v = u$ (given) $\therefore\ c = 1/u$

$\therefore$ From (i) we get $\dfrac{1}{v} = ks + \dfrac{1}{u} = \dfrac{ksu + 1}{u}$

or $$v = u/(ksu + 1)$$

or $$\frac{ds}{dt} = \frac{u}{ksu + 1} \quad \text{or} \quad \left(\frac{ksu + 1}{u}\right) ds = dt$$

Integrating, $$\frac{(1 + ksu)^2}{2ku^2} = t + c_2, \qquad \text{...(ii)}$$

where c_1 is constant of integration.

Initially $s = 0,\ t = 0 \quad \therefore\ c_1 = 1/(2ku^2)$

$\therefore$ From (ii) we get $$\frac{(1 + ksu)^2}{2ku^2} = t + \frac{1}{2ku^2}$$

or $$1 + 2kus + k^2u^2s^2 = 2ku^2t + 1$$

or $$t = (2kus + k^2u^2s^2)/(2ku^2)$$

or $$t = (s/u) + \tfrac{1}{2}ks^2.$$

Example 7. A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ time the cube of the velocity. Show that the distance it has described in time t is $1/\mu V)\ [\sqrt{(1 + 2\mu V^2 t)} - 1]$ and that its velocity then is $V/\sqrt{(1 + 2\mu V^2 t)}$. *(Gorakhpur 2006)*

Solution. The equation of motion of the particle is

$$m\frac{dv}{dt} = m\mu v^3$$

or $$\frac{dv}{v^3} = -\mu\, dt$$

Integrating, $-(1/2v^2) = -\mu t + C$, where C is constant of integration.

Initially $v = V, t = 0, \quad \therefore\ C = -1/2V^2$

$\therefore$ $$-\frac{1}{2v^2} = -\mu t - \frac{1}{2V^2}$$

or $$\frac{1}{v^2} = 2\mu t + \frac{1}{V^2} = \frac{2\mu V^2 t + 1}{V^2}$$

or $$v^2 = \frac{V^2}{1 + 2\mu V^2 t}$$

or $$v = \frac{V}{\sqrt{(1 + 2\mu V^2 t)}}$$

or $$\frac{dx}{dt} = \frac{V}{\sqrt{(1 + 2\mu V^2 t)}}$$

or $$dx = \frac{V}{\sqrt{(1 + 2\mu V^2 t)}}\, dt$$

Integrating we have $x = (1/\mu V)\sqrt{(1 + 2\mu V^2 t)} + C_1, \qquad$ ...(i)

where C_1 is constant of integration.

Initially $x = 0,\ t = 0,\quad \therefore\ C_1' = -1/(\mu V)$

$\therefore$ From (i) we have $x = \dfrac{1}{\mu V}\left[\sqrt{(1 + 2\mu V^2 t)} - 1\right].$

Example 8. A heavy particle is projected upwards in a medium the resistance of which varies as the square of the velocity. It has a kinetic energy K in its upward path at a given point, when it passes the same point on the way down, show that its loss of energy is $\dfrac{K^2}{K' + K}$, where K' is the limit to which the energy approaches in its downward course.

Solution. When the particle is going upwards, the equation of motion is

$$m\frac{d^2x}{dt^2} = -mg - m\mu v^2$$

or
$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{\mu}{g}v^2\right) \quad \text{...(i)}$$

When the particle is coming downwards, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - m\mu v^2$$

or
$$\frac{d^2x}{dt^2} = g\left(1 - \frac{\mu}{g}v^2\right) \quad \text{...(ii)}$$

$\therefore$ If V be terminal velocity of the particle, then the from (ii) we get

$$1 - (\mu/g)V^2 = 0 \quad \text{or} \quad V^2 = g/\mu \quad \text{...(iii)}$$

$\therefore$ From (i) we have

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right)$$

or
$$v\frac{dv}{dx} = -g\left(\frac{v^2 + V^2}{V^2}\right), \qquad \because\ \frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

or
$$\frac{2v\,.\,dv}{v^2 + V^2} = -\frac{2g}{V^2}dx$$

Integrating we get $\log(v^2 + V^2) = -(2g/V^2)x + c,$ ...(v)

where c is constant of integration.

Initially $x = 0$ and let $v = u$. Then $c = \log(u^2 + V^2)$

So from (iv)
$$\frac{2g}{V^2} = \log\left(\frac{u^2 + V^2}{v^2 + V^2}\right) \quad \text{...(iv)}$$

Let the height of the given point above the point of projection be h. Let v_1 be the velocity of the particle at this point while going upwards. Then from (v) we get

$$\frac{2gh}{V^2} = \log\left\{\frac{u^2 + V^2}{v_1^2 + V^2}\right\} \quad \text{...(vi)}$$

Let H be the greatest height upto which the particle can rise, then at $x = H, v = 0.$

∴ From (v) we get

$$\frac{2gH}{V^2} = \log\left\{\frac{u^2 + V^2}{V^2}\right\} \quad \text{...(vii)}$$

Again from (ii) we have

$$v\frac{dv}{dx} = g\left\{1 - \frac{\mu}{g}v^2\right\}, \qquad \because \frac{d^2x}{dt^2} = \frac{v\,dv}{dx}$$

or $$v\frac{dv}{dx} = g\left\{1 - \frac{v^2}{V^2}\right\} \quad \text{or} \quad \frac{2v\,dv}{V^2 - v^2} = \frac{2g}{V^2}dx$$

Integrating we have

$$(2g/V^2)\,x = -\log(V^2 - v^2) + c_1, \quad \text{...(viii)}$$

where c_1 is constant of integration.

At the highest point $x = 0$ [∵ for the equation (ii) x is being measured from the highest poing] and $v = 0$, ∴ $c_1 = \log V^2$

∴ From (viii) we get

$$(2g/V^2)\,x = \log V^2 - \log(V^2 - v^2)$$

or $$\frac{2g}{V^2}x = \log\left\{\frac{V^2}{V^2 - v^2}\right\} \quad \text{...(ix)}$$

Now the depth of the given point below the highest point is $H - h$. Let v_2 be the velocity of the particle at the given point while coming downwards.

Then from (ix) we get

$$\frac{2g}{V^2}(H - h) = \log\left\{\frac{V^2}{V^2 - v_2^2}\right\} \quad \text{...(x)}$$

or $$\frac{2gH}{V^2} - \frac{2hg}{V^2} = \log\left\{\frac{V^2}{V^2 - v_2^2}\right\}$$

or $$\log\left\{\frac{u^2 + V^2}{V^2}\right\} - \log\left\{\frac{u^2 + V^2}{v_1^2 + V^2}\right\} = \log\left\{\frac{V^2}{V^2 - v_2^2}\right\},$$

from (vi) and (vii)

or $$\log\left\{\frac{v_1^2 + V^2}{V^2}\right\} = \log\frac{V^2}{V^2 - v_2^2}$$

or $$\frac{v_1^2 + V^2}{V^2} = \frac{V^2}{V^2 - v_2^2}$$

or $$(v_1^2 + V^2)(V^2 - v_2^2) = V^4$$

or $$(v_1^2 - v_2^2)V^2 = v_1^2 v_2^2 \quad \text{or} \quad v_2^2(v_1^2 + V^2) = v_1^2 V^2$$

or $$v_2^2 = v_1^2 V^2/(v_1^2 + V^2) \quad \text{...(xi)}$$

∴ The required loss in energy $= \frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2$