12. Maxima and Minima of Function of Two Variables

12.1. Maxima

A function $f(x, y)$ is called **maximum** at the point $x = a$ and $y = b$ *i.e.*, at the point (a, b) if value of the function $f(x, y)$ is smaller than the value of $f(a, b)$ for all points (x, y) in the small neighbourhood of (a, b) except the point (a, b), *i.e.*,

$$f(x, y) < f(a, b).$$

Definition : Let $f(x, y)$ be a function of two independent variables x, y such that it is continuous and finite in the neighbourhood of their values a and b (say) respectively. Then the value of $f(a, b)$ is called **maxima** if

$$f(a + h, b + k) < f(a, b).$$

Whether be the relative value of h and k, positive or negative, provided h and k are finite and sufficiently small.

12.2. Minima

A function $f(x, y)$ is called **minimum** at the point $x = a$ and $y = b$, *i.e.*, at the point (a, b) if value of the function $f(x, y)$ is greater than the value of $f(a, b)$ for all points (x, y) in the small neighbourhood of (a, b) except the point (a, b), *i.e.*,

$$f(x, y) > f(a, b).$$

Definition : Let $f(x, y)$ be a function of two independent variables x, y such that it is continuous and finite in the neighbourhood of their value a and b (say) respectively. Then the value of $f(a, b)$ is called **minima** if

$$f(a + h, b + k) > f(a, b).$$

Whether be the relative value of h and k, positive or negative, provided h and k are finite and sufficiently small.

13. Stationary and Extreme Points

Definition : A point (a, b) is called a **stationary point** if all the first order partials derivatives of the function vanish at that point. A *stationary point*, at which function is a maximum or a minimum is known as extreme point and the value of the function at extreme point is known as *extreme value.*

From above we concude that a stationary point may be a maximum or a minimum or neither of these two.

14. Local Extreme Value

14.1. Local Maximum

Definition : Let $z = f(x, y)$ be a function defined in the circular region D of R^2 then f has a **local maximum** at (a, b) if circular region D containing (a, b) is such that

$$f(x, y) \leq f(a, b) \ \forall \ (x, y) \in D \subseteq R^2.$$

14.2. Local Minimum

Definition : Let $z = f(x, y)$ be a function defined in the circular region D of R^2 then f has a local minimum at (a, b) if circular region D containing (a, b) is such that

$$f(x, y) \geq (a, b) \ \forall \ (x, y) \in D \subseteq R^2.$$

15. Necessary Condition for the Existence of Maxima or Minima at a Point

Theorem : *A necessary condition for $f(x, y)$ to have an extreme value (maximum or minimum) at (a, b) is that $f_x(a, b) = 0$ and $f_y(a, b) = 0$, provided these partial derivatives exist.*

Proof : To determine maxima or minima of a function $f(x, y)$ at a point (a, b), we investigate the sign of

$$f(a + h, b + k) - f(a, b) \qquad ...(1)$$

which has invariable sign for all sufficiently small and finite values of h and k, positive or negative.

By Taylor's theorem,

$$f(a + h, b + k) = f(a, b) + \left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right)_{\substack{x = a \\ y = b}}$$

$$+ \frac{1}{2!}\left(h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \, \partial y} + k^2 \frac{\partial^2 f}{\partial y^2}\right)_{\binom{x = a}{y = b}} + \dots \qquad ...(2)$$

Now, for small values of h and k the second and higher order terms are much smaller numerically than the first order terms, and may be disregarded when determining the sign of (1). Thus, the sign of (1) depends upon the first degree expression

$$\left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right)_{(a, b)}$$

But this expression change their sign when the sign of h and k are changed. Hence, in order that (1) may preserve a fixed sign it is necessary that

$$\left(h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y}\right)_{\binom{x = a}{y = b}} = 0 \qquad ...(3)$$

Since h and k are independent and non-zero. Therefore, (3) implies that

$$\left(\frac{\partial f}{\partial x}\right)_{(a, b)} = f_x(a, 0) = 0$$

and

$$\left(\frac{\partial f}{\partial y}\right)_{(a, b)} = f_y(a, b) = 0.$$

Hence, the necessary conditions that $f(x, y)$ should have a maximum or minimum at (a, b) are that

$$f_x(a,b) = 0, f_y(a,b) = 0.$$

Note : The function $f(x,y) = |x| + |y|$ has an extreme value (minimum) at (0, 0) even though the partial derivatives f_x and f_y do not exist at (0, 0).

16. Sufficient Conditions for $f(x,y)$ to have an Extreme Value (Maxima and Minima) at (a,b) (Lagrange's Condition)

Let $f(x,y)$ possesses continuous second order partial derivatives in a certain neighbourhood of (a,b) and these derivatives at (a,b), *viz.*, $f_{xy}(a,b), f_{xx}(a,b), f_{yy}(a,b)$ are not all zero. Then by Taylor's theorem, we have

$$f(a+h, b+k) = f(a,b) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right)_{(a,b)}$$

$$+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x\,\partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right)_{(a,b)} + R_3,$$

where R_3 consists of terms of third and higher order in h and k.

But necessary conditions for maximum or minimum at (a,b) are satisfied, *i.e.*,

$$f_x(a,b) = 0 = f_y(a,b).$$

Therefore,

$$f(a+h, b+k) - f(a,b) = \frac{1}{2!}(rh^2 + 2hks + tk^2) + R_3 \qquad ...(1)$$

where

$$r = f_{xx}(a,b), s = f_{xy}(a,b), t = f_{yy}(a,b)$$

Now, by taking h and k sufficiently small, the second degree terms in R.H.S. of (1) may be made to govern the sign of right hand side and therefore, of the L.H.S. also.

Thus, we can write

$$rh^2 + 2hks + tk^2 = \frac{1}{r}[r^2h^2 + 2hkrs + rtk^2]$$

$$= \frac{1}{r}[r^2h^2 + 2hkrs + s^2k^2 + rtk^2 - s^2k^2]$$

$$= \frac{1}{r}[(rh + sk)^2 + k^2(rt - s^2)] \qquad ...(2)$$

The first term inside the brackets is positive. The second will also be positive if $rt - s^2 > 0$. Thus, the expression (2) will have the same sign as r, for all values of h and k. This sign is determined by the sign of r. Now, we consider following cases.

Case I. If $rt - s^2 > 0$ and $r > 0$, then

$$f(a+h, b+k) - f(a,b) > 0,$$

for all small values of h and k. Hence, $f(x,y)$ has a maxima at (a,b).

Case II. If $rt - s^2 > 0$ and $r < 0$, then

$$f(a+h, b+k) - f(a, b) < 0.$$

Hence, $f(x, y)$ has a minima at (a, b).

Case III. If $rt - s^2 = 0$, then further investigation is needed to determine whether $f(x, y)$ is a maximum or minimum at $x = a, y = b$ or not.

Case IV. Saddle Points : If $rt - s^2 < 0$, then $f(x, y)$ has neither a maxima nor minima at (a, b). In this case $f(a+h, b+k) - f(a, b)$ is not of invariable sign, *i.e.*, it has one sign for some values of h, k while it has another sign for other values of h, k. Such point is called a saddle point.

17. Definitions : Stationary Points and Stationary Values

A point (a, b) is called a stationary point if first order partial derivatives of the function $f(x, y)$ vanish at that point. Thus, if $f(x, y)$ is a function of two independent variables, then

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

Now at the stationary points, $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$. Therefore,

$$df = 0.$$

Hence, stationary points can be obtained by solving following equations simultaneously :

$$\left(\frac{\partial f}{\partial x}\right) = 0$$

and

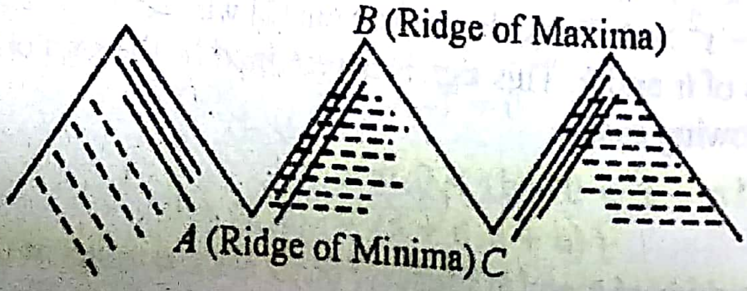
$$\left(\frac{\partial f}{\partial y}\right) = 0.$$

The value of the function obtained at stationary points are called stationary values.

Note : Extreme points, the points of "ridge" of maximum (or minimum) saddle point are classified as stationary points.

18. Ridge of Maximum (or Minimum)

If the surface falls (or rises) in all directions except that of the ridge where it remains stationary is called 'ridge' of maximum (or minimum). Thus, we can explain it geometrically as,



9. Working Rule for Determining the Maxima and Minima of Function of Two Variables

(1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, and equate them to zero.

(2) Solve these simultaneous equations $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ for x and y. Then pairs of values of x and y, thus obtained will give stationary values of $f(x, y)$. Let (a, b) be one of the pair of roots.

(3) Find $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2}$ and substitute the point (a, b) in these. Calculate $rt - s^2$ for the point (a, b).

(4) (i) If $(rt - s^2) > 0$ and $r < 0$, then $f(x, y)$ has a maximum at (a, b) and this point is called point of maxima.

(ii) If $(rt - s^2) > 0$ and $r > 0$, then $f(x, y)$ has minimum at (a, b) and this point is called point of minima.

(iii) If $rt - s^2 < 0$, then $f(x, y)$ has neither maximum nor minimum at (a, b) and therefore function has saddle point there.

(iv) If $rt - s^2 = 0$, the case is undecided and further investigation is necessary to decide it.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the maximum or minimum values of the function*

$$x^3 y^2 (1 - x - y).$$

(Purvanchal 2005; Kumayun 2000; Gorakhpur 2002)

Solution : Let $u = x^3 y^2 (1 - x - y)$.

For maxima or minima, we have

$$\frac{\partial u}{\partial x} = 3x^2 y^2 (1 - x - y) - x^3 y^2 = 0 \quad \text{...(i)}$$

$$\frac{\partial u}{\partial y} = 2x^3 y (1 - x - y) - x^3 y^2 = 0 \quad \text{...(ii)}$$

Subtracting equation (ii) from (i), we get

$$x^2 y (1 - x - y)(3y - 2x) = 0$$

$$\therefore \quad y = \frac{2}{3}x$$

Putting the value of y in equation (i), we get

$$3x^2 \cdot \frac{4}{9}x^2 \left(1 - x - \frac{2}{3}x\right) - x^3 \cdot \frac{4}{9}x^2 = 0$$

$$\text{or} \quad \frac{4}{9}x^4 [3 - 5x - x] = 0$$

$$\text{or} \quad \frac{4}{9}x^4 (3 - 6x) = 0$$

Maximum value of V = $\frac{\;}{3}\cdot\frac{\;}{3}(2)(24-8-8) = \frac{\;}{9}$ cube unit.

Example 7. *Discuss the maximum or minimum values of* $u = x^2y^2 - 5x^2 - 5y^2 - 8xy.$ **(Avadh 2006)**

Solution : We have given

$$u = x^2y^2 - 5x^2 - 5y^2 - 8xy.$$

$$\therefore \quad \frac{\partial u}{\partial x} = 2xy^2 - 10x - 8y$$

and $$\frac{\partial u}{\partial y} = 2x^2y - 10y - 8x$$

Thus for maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = 2xy^2 - 10x - 8y = 0 \qquad ...(i)$$

and $$\frac{\partial u}{\partial y} = 2x^2y - 10y - 8x = 0 \qquad ...(ii)$$

Subtracting equation (i) by equation (ii), we have

$$2xy(x-y) + 10(x-y) - 8(x-y) = 0$$

$$\Rightarrow \quad 2xy(x-y) + 2(x-y) = 0$$

$$\Rightarrow \quad (x-y)(xy+1) = 0$$

$$\Rightarrow \quad y = x \text{ or } y = -\frac{1}{x}.$$

When $y = x$, we have from (i)

$$2x^3 - 18x = 0$$

$$\Rightarrow \quad x(x^2 - 9) = 0$$

$$\Rightarrow \quad x = 0, x = \pm 3.$$

When $x = 0, y = 0$, when $x = \pm 3, y = \pm 3.$

And when $y = -\frac{1}{x}$, equation (1) gives

$$\frac{2}{x} - 10 + \frac{8}{x} = 0$$

$$\Rightarrow \quad 1 - x^2 = 0 \Rightarrow x = \pm 1$$

When $x = 1, y = -1$, when $x = -1, y = 1$.

Hence, (0, 0), (3, 3), (−3, −3), (1, −1), (−1, 1) are critical points of u.

Now $\quad r = \frac{\partial^2 u}{\partial x^2} = 2y^2 - 10, s = \frac{\partial^2 u}{\partial x \partial y} = 4xy - 8$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^2 - 10.$$

Point	(0, 0)	(± 3, ± 3)	(1, −1)	(−1, 1)
$r =$	−10	8	−8	−8
$s =$	−10	26	−12	−12
$t =$	−10	8	−8	−8
$rt - s^2 =$	36	$64 - 28^2$ = −ve	64 − 144 = −80 = −ve	64 − 144 = −80 = −ve
Result	Maxima	Neither maxima nor minima	Neither maxima nor minima	Neither maxima nor minima

Hence, at only (0, 0), we get maximum value of u.