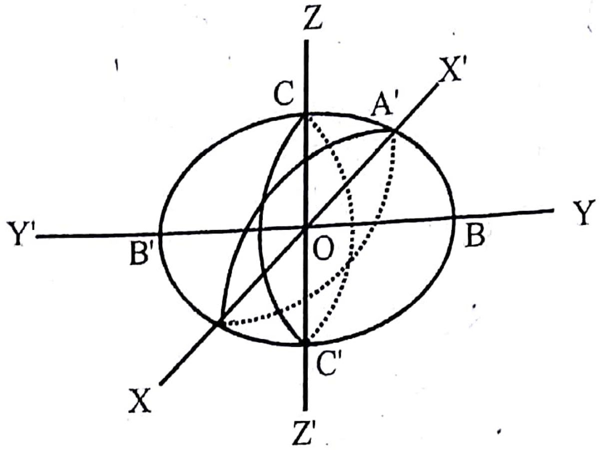
8 Central Conicoids

Key at a Glance

- **Ellipsoid** : The standard equation of the ellipsoid is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...... (1)}$$

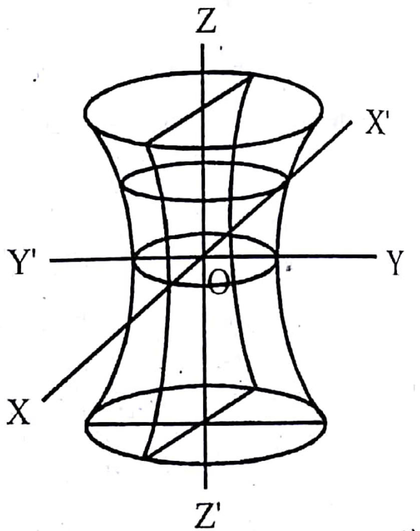
origin is the centre of this ellipsoid & the cordinate planes bisects all chords perpendicular to them. Figure of the ellipsoid is as shown below :



- **Hyperboloid of one sheet** : The standard equations on hyerboloid of one sheet is given by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

origin is the centre of this hyperboliod of one sheet & the coordinate planes bisect all chords perpendicular to them. The figure of hyperboliod of one sheet is as given below :

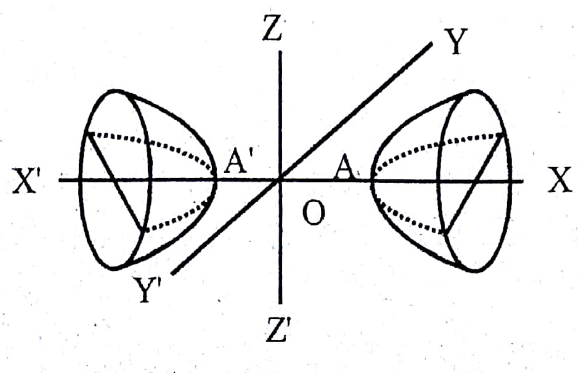


(Hyperboloid of one sheet)

- **Hyperboloid of two sheet :** The standard equation of hyperboloid of two shet is given by

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

origin is the centre of this hyperboloid of two sheets, the coordinate planes bisect all chords perpendicular to them. The figure of hyerboloid of two sheets is as given below :



(Hyperboloid of two sheets)

- **Central Coincoid** : The combined name of ellipsoid, hyperboloid of one sheet & hyperboloid of two sheets is central coincoid. The standard equation of central coincoid is

$$ax^2 + by^2 + cz^2 = 1$$

(i) It represents an ellipsoid if a,b,c are all positive
(ii) it represents a hyperboloid of one sheet if any two of a,b,c are positive & the remaining third is negative
(iii) It represents a hyberboliod of two sheets. if any two of a,b,c are negative & the remaining third is positive.

- **Tangent Plane** : Equation of tangent plane at the point (x_1, y_1, z_1) of the conicoid $ax^2 + by^2 + cz^2 = 1$ is $axx_1 + byy_1 + czz_1 = 1$
- **Condition of Tangent** : The plane $lx + my + nz = p$ touch the conicoid $ax^2 + by^2 + cz^2 = 1$ if

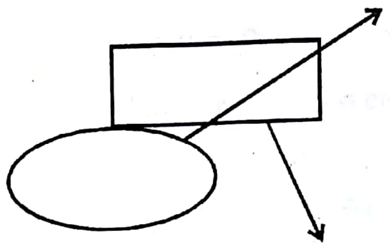
$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

- **Polar Plane** : The equation of polar plane of point (x_1, y_1, z_1) with respect to conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$T \equiv axx_1 + byy_1 + czz_1 - 1 = 0$$

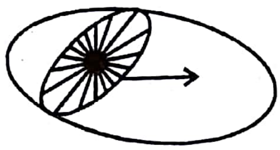
- **Normal :** Equation of the normal to the conicoid $ax^2 + by^2 + cz^2 = 1$ at the point (α, β, γ) is

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\upsilon}$$

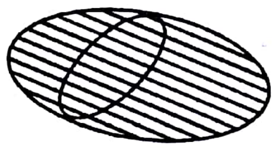


- **Diametral Plane** : The diametral plane of a conicoid is the locus of mid point of parallel chords.

If the chords are parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$,
the diametral plane of the conicoid $ax^2 + by^2 + cz^2 = 1$ is $alx + bmy + cnz = 0$



Section with given Centre



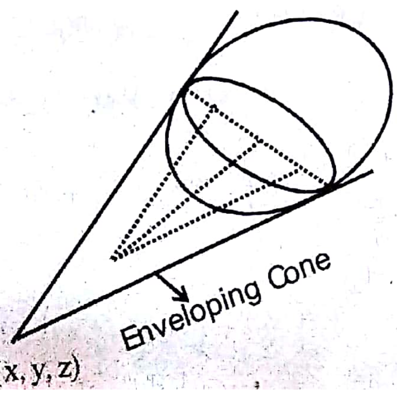
Diametral Plane

- **Section with given centre** : If $ax^2 + by^2 + cz^2 = 1$ be equation of given conicoid then plane section of the concoid whose centre is the point (x_1, y_1, z_1) is-

$$T = S_1 \quad \text{where} \quad T = axx_1 + byy_1 + czz_1 - 1$$

$$S_1 = ax_1^2 + by_1^2 + cz_1^2 - 1$$

- **Enveloping cone of the conicoid** : Equation of enveloping cone of the conicoid $S_1 = ax^2 + by^2 + cz^2 - 1 = 0$ is $SS_1 = T^2$



Q. 4 If PQR be te extremities of three conjugate semi diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ then prove that pole of the plane lie on the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$

(2003,2004,2008)

OR

Prove that the pole of the plane PQR lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$

(2013)

OR

Let OP, OQ, OR be the conjugate semi-diameters of ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ **(2014,2016,201**

Find the equation of the plane PQR.

Sol^n : Let (α,β,γ) be the pole of the plane PQR w.r.t. the given ellipsoid

$\therefore$ Equation of the polar plane of (α,β,γ) ...... (1)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...... (2)$$

is
$$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}=1$$

Also the equation of the plane PQR is .... (3)

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1$$

where extremities $P=(x_1,y_1,z_1)$, $Q=(x_2,y_2,z_2)$ & $R=(x_3,y_3,z_3)$

Comparing (2) and (3), we get

$$\alpha=(x_1+x_2+x_3), \beta=(y_1+y_2+y_3), \gamma=(z_1+z_2+z_3)$$

Now, we have

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{(x_1+x_2+x_3)^2}{a^2}+\frac{(y_1+y_2+y_3)^2}{b^2}+\frac{(z_1+z_2+z_3)^2}{c^2}$$

$$=\frac{1}{a^2}\sum x_1^2+\frac{1}{b^2}\sum y_1^2+\frac{1}{c^2}\sum z_1^2+2\left\{\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}\right)+\left(\frac{x_1x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}\right)+\left(\frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}\right)\right\}$$

$$=\frac{a^2}{a^2}+\frac{b^2}{b^2}+\frac{c^2}{c^2}=3 \qquad \text{since } \sum x_1x_2=0 \text{ etc}$$

The locus of the pole (α,β,γ) is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$ **Proved.**

Q.5 Define the direction sphere of the central conicoid $ax^2+by^2+cz^2=1$ & also write its equation. **(2001,2006,2011)**

Sol^n : Director sphere : - The director spher of a central conicoid is the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid.

The equation of the central conicoid is

$$ax^2+by^2+cz^2=1 \quad ...... (1)$$

Let the equations of three tangent planes to (1) be

$$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad ...... (2)$$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad ...... (3)$$

and
$$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad ...... (4)$$

Where l_1,m_1,n_1 ; l_2,m_2,n_2 and l_3,m_3,n_3 are the actual direction cosines of the normals to the above threee tangent planes respectively. If these tangent planes are mutually perpendicular, then l_1,l_2,l_3 ; m_1,m_2,m_3 and n_1,n_2,n_3 are also dc's of any three mutually perpendicular lines.

Therefore, we have

$$l_1^2+l_2^2+l_3^2=1,\ m_1^2+m_2^2+m_3^2=1 \text{ and } n_1^2+n_2^2+n_3^2=1 \quad \text{...... (5)}$$

$$l_1m_1+l_2m_2+l_3m_3=0, m_1n_1+m_2n_2+m_3n_3=0, \ \& \ \ n_1l_1+n_2l_2+n_3l_3=0 \ \text{...} \quad (6)$$

The directior sphere of the surface (1), being the locus of the point of intersection of the three tangent planes (2),(3) and (4) is obtained by eliminating l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 between the equation (2),(3) and (4) with the help of the relation (5) and (6). Squaring and adding (2), (3) and (4), we get

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2=$$

$$\frac{1}{a}\sum l_1^2+\frac{1}{b}\sum m_1^2+\frac{1}{c}\sum n_1^2$$

or $$x^2\sum l_1^2+y^2\sum m_1^2+z^2\sum n_1^2+2yz\sum m_1n_1+2zx\sum n_1l_1+2xy\sum l_1m_1$$

$$=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad \text{(Using 5)}$$

Or, $$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \qquad \text{Using (5) and (6)}$$

Q. 14 Show that the plane $x+2y+3z=2$ touches the conicoid $x^2-2y^2+3z^2=2$

(2004)

Solⁿ : Let α,β,γ be the point of contact of the given plane with conicoid $x^2-2y^2+3z^2=2$. Then tangent plane at this point is $\alpha x-2\beta y+3\gamma z=2$ on comparing this with the given plane $x+2y+3z=2$, We get

$$\alpha=1,\beta=-1,\gamma=1$$

Thus the point of contact (1,-1,1) on putting this point in the equation of conicoid, we have

$$1-2+3=2 \quad\Rightarrow\quad 2=2$$

This show that the point lie on the conicoid & given plane is tangent plane at the point (1,-1,1).

Q.15 Prove that six normal can be drawn to an ellipsoid from a given point α,β,γ.

(DDU-2004,2010,2016,2018)

Solⁿ : Normal at a point (x_1,y_1,z_1) to the elliposoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2} \qquad \text{....(1)}$$

f the normal passes through α,β,γ then

$$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda \quad \text{(say)}$$

i.e.
$$x_1=\frac{a^2\alpha}{a^2+\lambda},y_1=\frac{b^2\beta}{b^2+\lambda},z_1=\frac{c^2\gamma}{c^2+\lambda}$$

As the point (x_1,y_1,z_1) lies on ellipsoid, we have

$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1$$

$$(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2-a^2\alpha^2(b^2+\lambda)^2-$$
$$b^2\beta^2(a^2+\lambda)^2(c^2+\lambda)^2-c^2\gamma^2(a^2+\lambda)^2(b^2+\lambda)^2=0$$

This is of degree of six in λ. So correspoinding to six value of λ. We have six point on the ellipsoid the normal at which passes through α,β,γ .Thus there are six normals that can be drawn from a point α,β,γ to the ellipsoid.

Q. 16 Find the value of 'a' so that the plane $ax+12y-6z=17$ touches the conicoid $3x^2-6y^2+9z^2+17=0$.

(DDU-2005,2017)

Solⁿ : Equation of the given plane is $ax+12y-6z=17$ ...... (1)

& conicoid is $3x^2-6y^2+9z^2+17=0$

$$\frac{3}{-17}x^2+\frac{6}{17}y^2+\frac{9}{-17}z^2=1 \qquad \text{...... (2)}$$

The plane (1) touches the conicoid (2), If

$$\frac{a^2}{3/-17}+\frac{(12)^2}{6/17}+\frac{(-6)^2}{9/-17}=(17)^2 \Rightarrow \frac{-a^2}{3}+\frac{144}{6}+\frac{36}{-9}=17$$

$$\frac{-a^2}{3}=17-20=-3\Rightarrow -a^2=-3\times 3=9=a^2=9\Rightarrow a=\pm 3$$

Thus is the required value of a is ± 3

... if the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ casts a circular

$ax + by + cz = 3(x + y + z)$

Q. 19 Prove that the sum of the squares of the projection of three conjugate semi-diameters on a line is constant. **(2006)(SU-2016,2017)**

Sol^n : Proof : Let the direction cosins of a given line be l,m,n. Let OP, OQ and OR be the conjugate semi-diameters of an ellipsoid.

The projection of OP on the line where d.c's are l,m,n

$$= (x_1 - 0)l + (y_1 - 0)m + (z_1 - 0)n = lx_1 + my_1 + nz_1$$

Similarly the projection of OQ and OR in the given line are $lx_2 + my_2 + nz_2$ and $lx_3 + my_3 + nz_3$ respectively.

The sum of the square of the projections of OP,OQ and OR on the given line

$$= (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$

$$= l^2(x_1^2 + x_2^2 + x_3^3) + m^2\sum y_1^2 + n^2\sum z_1^2 + 2lm\sum x_1y_1 + 2mn\sum y_1z_1 + 2nl\sum z_1x_1$$

$$= l^2a^2 + m^2b^2 + n^2c^2 + 0 + 0 + 0 \Rightarrow l^2a^2 + m^2b^2 + n^2c^2$$

Which is a constant.

The required equation of plane is $T = S_1$

or $$\frac{-3}{2}x + \frac{3}{2}y + \frac{15}{4}z - 1 = \frac{1}{2} \Rightarrow 3x - 3y - 5z + 2 = 0$$

Q. 21 Find the locus of straight line drawn through a fixed point (α, β, γ) at the right angles to their polar with respect to the conicoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ **(DDU-2008,2018)**

Solⁿ : The equation of the given conicoid is

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \quad \text{....... (1)}$$

The equation of any straight line through the point (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \quad \text{...... (2)}$$

The co- ordinates of any point on (2) are $(lr+\alpha, mr+\beta, nr+\gamma)$.

The equation of the polar plane of this point w.r.t. to conicoid (1) is

$$ax(lr+\alpha) + by(mr+\beta) + cz(nr+\gamma) = 1$$

or $$(ax\alpha + by\beta + cz\gamma - 1) + r(axl + bym + czn) = 0$$

This plane for all values of r will pass through the line

$$ax\alpha + by\beta + cz\gamma - 1 = 0, \quad axl + bym + czn = 0 \quad \text{...... (3)}$$

This is the polar of the line (2) w.r.t. the conicoid (1)

Let λ, μ, υ be the direction ratios of the polar line (3), so that we have

$$a\alpha\lambda + b\beta\mu + c\gamma\upsilon = 0, axl + bym + czn = 0$$

Solving, $$\frac{\lambda}{bc(n\beta - m\gamma)} = \frac{\mu}{ca(l\gamma - n\alpha)} = \frac{\upsilon}{ab(m\alpha - l\beta)}$$

Since the lines (2) and (3) ar given to tbe perpendicular, we have

$$l\lambda + m\mu + n\upsilon = 0$$

$$lbc(n\beta - m\gamma) + mca(l\gamma - n\alpha) + nab(m\alpha - l\beta) = 0$$

or, dividing by $abclmn$ throughout, we get

$$\frac{\beta}{am} - \frac{\gamma}{an} + \frac{\gamma}{bn} - \frac{\alpha}{bl} + \frac{\alpha}{cl} - \frac{\beta}{cm} = 0$$

or, $$\frac{\alpha}{l}\left(\frac{1}{c} - \frac{1}{b}\right) + \frac{\beta}{m}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{\gamma}{n}\left(\frac{1}{b} - \frac{1}{a}\right) = 0$$

The locus of the line (2) is obtained by eliminating l, m, n between (2) and (5) and hence is given by

$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{y-\beta}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{z-\gamma}\left(\frac{1}{a} - \frac{1}{b}\right) = 0$$

Q. 22 Show that plane $x + 2y + 3z = 2$ touches the conicoid $x^2 + 2y^2 + 3z^2 = 2$ **(2009)**

Solⁿ : Given conicoid $x^2 + 2y^2 + 3z^2 = 2$

or, $\frac{x^2}{2} - y^2 + \frac{3}{2}z^2 = 1$ ...... (1)

equation of the given plane is $x + 2y + 3z = 2$ ...... (2)

If the plane (2) touches concicoid (1) then applying condition

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

$$\frac{1}{1/2} + \frac{4}{-1} + \frac{9}{3/2} = 4 \quad \text{Or } 2 - 4 + 6 = 4 \quad \text{or } 4 = 4$$

Hence, plane (2) touches conicoid (1)

Q.23 Find the equation of the normal to a central conicoid $ax^2 + by^2 + cz^2 = 1$ at (α, β, γ). **(2009)**

Sol^n : Let, the equation of central conicoid be

$$ax^2 + by^2 + cz^2 = 1 \quad \text{...... (1)}$$

The equation of the tangent plane at (α, β, γ) of the conicoid equ^n (1) is

$$a\alpha x + b\beta y + c\gamma z = 1$$

The normal to the conicoid equ^n(1) at (α, β, γ) is the straight line perpendicular to the tangent plane equ^n(2) and passing through (α, β, γ), and hence the required equations of the normal are given by

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$

Where, $a\alpha, b\beta, c\gamma$ are the direction ratios of the normal equ^n (3)

Now let p be the length of the perpendicular from the origin to the tangent plane equ^n (2) so that

$$p = \frac{1}{\sqrt{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2}} \quad \text{or}$$

$$(a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1 \quad \text{...... (4)}$$

In view of equ^n(4) the actual direction cosines of the normal equ^n (3) are $a\alpha p, b\beta p, c\gamma p$ and hence the equ^n(3) of the normal to the conicoid equ^n(1) at (α, β, γ) in terms of actual direction cosines are given by

$$\frac{x-\alpha}{a\alpha p} = \frac{y-\beta}{b\beta p} = \frac{z-\gamma}{c\gamma p}$$ **Proved.**

Q.24 Find the equation of envelopoing cone of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ **(2009,2014)**

Sol^n : Let, the equation ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{....... (1)}$$

Suppose $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r\ (say)$ be the equation of a line passing through a given point (α, β, γ).

Any point on the line is $(lr + \alpha, mr + \beta, nr + \gamma)$

Suppose the line meets the conicoid at this point, then

$$\frac{(lr+\alpha)^2}{a^2} + \frac{(mr+\beta)^2}{b^2} + \frac{(nr+\gamma)^2}{c^2} = 1 \quad \text{...... (2)}$$

This is quadratic in r, so the line meets the conicoid in two point. The line will be tangent to the ellipsoid if these two points are coincident i.e. two roots of r from (2) must be equal

$$4\left(\frac{l\alpha}{a^2}+\frac{m\beta}{b^2}+\frac{n\gamma}{c^2}\right)^2=4\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right) \quad \text{...... (3)}$$

The locus of tangent line to the ellipsoid can be obtained by eleminating l,m,n between (1) and (3)

$$\left(\frac{(x-\alpha)\alpha}{a^2}+\frac{(y-\beta)\beta}{b^2}+\frac{(z-\gamma)\gamma}{c^2}\right)=\left(\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}+\frac{(z-\gamma)^2}{c^2}\right)$$

$$\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right) \quad \text{.......... (4)}$$

If $S=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$ $\quad S_1=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1$

and $T=\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1$

Then (4) can be written as $SS_1=T^2$

i.e. $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}-1\right)^2$$

which is required enveloping cone.

Q.25 Defined conjugate diameters and conjugate diametrol planes of an ellipsoids. Show that $OP^2+OQ^2+OR^2=\text{constant}$ **(2009)**

Solⁿ : Conjugate semi- diameters : The three semi-diameters OP,OQ and OR of an ellipsoid which are such that the plane containing any two is the diametral plane of the third are called the conjugate semi-diameters.

Conjugate Planes : The three planes POQ, QOR and ROP , which are such that each is the diametral plane of the line of intersection of the other two, are called the conjugate planes or conjugate diametral planes.

Property : The sum of square of any three conjugate semi-diameters of ellipsoid is constant.

We have $OP^2+OQ^2+OR^2$

$$=\left(x_1^2+y_1^2+z_1^2\right)+\left(x_2^2+y_2^2+z_2^2\right)+\left(x_3^2+y_3^2+z_3^2\right)$$

or, $$=\left(x_1^2+x_2^2+x_3^2\right)+\left(y_1^2+y_2^2+y_3^2\right)+\left(z_1^2+z_2^2+z_3^2\right)$$

$$=a^2+b^2+c^2 = \text{constant.}$$

Q.26 Let OP, OQ, OR be the conjugate semidiameters of $3x^2+y^2+\lambda z^2=1$. Find λ If $OP^2+OQ^2+OR^2=2$ **(2010)**

Solⁿ : Equation of central conicoid is

$$3x^2+y^2+\lambda z^2=1$$

or, $$\frac{x^2}{1/3}+\frac{y^2}{1}+\frac{z^2}{1/\lambda}=1$$

We know that $OP^2+OQ^2+OR^2=a^2+b^2+c^2$

$$2=\frac{1}{3}+1+\frac{1}{\lambda} \quad \text{or} \quad 1-\frac{1}{3}=\frac{1}{\lambda} \quad \text{or,} \quad \frac{2}{3}=\frac{1}{\lambda}$$

or $$\lambda=\frac{3}{2}$$

Q.28 If section of envloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose vertex is P by the plane z = 0 is a rectangular hyperbola ; show that locus of P is

$$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$$

(DDU-2013,2015,2018)

Sol^n = Let $P(\alpha,\beta,\gamma)$ be the co-ordinates of vertex.

The enveloping cone of ellepsoid is given by

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\lambda^2}{c^2}-1\right)=\left[\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\lambda z}{c^2}-1\right]^2$$

the section of this cone by the plane z= 0 is given by

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\lambda^2}{c^2}-1\right)=\left(\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}-1\right)^2$$

or

$$\frac{x^2}{a^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)+\frac{y^2}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)-\frac{2xy\alpha\beta}{a^2b^2}$$

$$+\frac{2\alpha x}{a^2}+\frac{2\beta y}{b^2}-\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)=0$$

Above eq^n will represent a rectangular hyperbola if

coeff of x^2 + coeff of $y^2 = 0$

$$\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}+\frac{\gamma^2}{c^2}-1\right)=0$$

$$\frac{\alpha^2}{a^2b^2}+\frac{\beta^2}{a^2b^2}+\frac{\gamma^2}{a^2c^2}+\frac{\gamma^2}{b^2c^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

$$\frac{\alpha^2+\beta^2}{a^2b^2}+\frac{\gamma^2}{c^2}\left[\frac{1}{a^2}+\frac{1}{b^2}\right]=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$

dividing by $\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$

$$\frac{\alpha^2+\beta^2}{a^2+b^2}+\frac{\gamma^2}{c^2}=1$$

taking locus of (α,β,γ)

$$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1$$

Q. 29 What is Director sphere ? Find the equation of Director sphere of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **(2011,2015)(SU-2018)**

Sol^n : Director sphere is locus of points of intersection of three mutually perpendicular tangent plane to the central conicoid.

Let the three mutually perpendicular tangent planes to ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ be

$$l_1x+m_1y+n_1z=\sqrt{a^2l_1^2+b^2m_1^2+c^2n_1^2} \quad \text{...... (i)}$$

$$l_2x+m_2y+n_2z=\sqrt{a^2l_2^2+b^2m_2^2+c^2n_2^2} \quad \text{...... (ii)}$$

$$l_3x+m_3y+n_3z=\sqrt{a^2l_3^2+b^2m_3^2+c^2n_3^2} \quad \text{........ (iii)}$$

Squaring and adding we get -

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2=$$
$$\left(a^2l_1^2+b^2m_1^2+c^2n_1^2\right)+\left(a^2l_2^2+b^2m_2^2+c^2n_2^2\right)+\left(a^2l_3^2+b^2m_3^2+c^2n_3^2\right)$$

or, $$x^2\left(l_1^2+l_2^2+l_3^2\right)+y^2\left(m_1^2+m_2^2+m_3^2\right)+z^2\left(n_1^2+n_2^2+n_3^2\right)$$
$$2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(m_1n_1+m_2n_2+m_3n_3)++2zx(n_1l_1+n_2l_2+n_3l_3)$$
$$=a^2\left(l_1^2+l_2^2+l_3^2\right)+b^2\left(m_1^2+m_2^2+m_3^2\right)+c^2\left(n_1^2+n_2^2+n_3^2\right)$$

Since, the three tangent plane are perdicular, therefore three normals asr also perpendicular. Using $l_1^2+l_2^2+l_3^2=1$, $l_1m_1+l_2m_2+l_3m_3=0$ etc.

$$x^2.1+y^2.1+z^2.1+0+0+0=a^2.1+b^2.1+c^2.1$$

$$\Rightarrow \quad x^2+y^2+z^2=a^2+b^2+c^2$$

Ans.

Q.30 : If OP,OQ,OR are the conjugate semi diamaters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Proved that the locus of the centre of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane PQ?R is the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$

(2011)(SU-2018)

Sol^n : Let the coordinate of P,Q and R are $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) respectively. Since OP, OQ,OR are the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{..... (1)}$$

Therefore equation of plane PQR is -

$$\frac{x_1+x_2+x_3}{a^2}+\frac{y_1+y_2+y_3}{b^2}+\frac{z_1+z_2+z_3}{c^2}=1 \quad \text{..... (2)}$$

Let (α,β,γ) be the centre of the section of the ellipsoid by the plane (ii). Then equation of the section is -

$$T=S_1 \quad \text{i.e.,}$$

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad \text{...... (3)}$$

Comparing equation (ii) and (iii) we get

$$\frac{\alpha}{x_1+x_2+x_3}=\frac{\beta}{y_1+y_2+y_3}=\frac{\gamma}{z_1+z_2+z_3}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\lambda(say)$$

Then $\alpha=\lambda(x_1+x_2+x_3)$, $\beta=\lambda(y_1+y_2+y_3)$, $\gamma=\lambda(z_1+z_2+z_3)$

and $$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\lambda \quad \text{....... (4)}$$

$$\therefore \quad \frac{\lambda^2(x_1+x_2+x_3)^2}{a^2}+\frac{\lambda^2(y_1+y_2+y_3)^2}{b^2}+\frac{\lambda^2(z_1+z_2+z_3)}{c^2}=\lambda$$

or $$\lambda\left[\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)+\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}\right)+\left(\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}\right)\right]$$
$$+2\left(\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}\right)+2\left(\frac{x_1x_3}{a^2}+\frac{y_1y_3}{b^2}+\frac{z_1z_3}{c^2}\right)$$
$$+2\left(\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}\right)=1$$

$$\Rightarrow \quad \lambda\,(1+1+1+2.0+2.0+2.0)=1 \Rightarrow \lambda=\frac{1}{3}$$

Form (4) we get -

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{1}{3}$$

$\therefore$ Locus of (α,β,γ) is -

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$$

Proved.

Q.31 : Define polar plane of a central conicoid and obtain its equation also.

(2013,2015)(SU-2016)

Sol^n : Part I) - Let $ax^2+by^2+cz^2=1 \quad \text{....... (1)}$

or, $2x+6y+15z-59=0$ **Ans.**

Q.35 Find the condition that the plane $lx+my+nz=p$ may touch the ellepsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **(DDU-2017)**

Solⁿ : The equation of the ellipsoid is-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{.......... (i)}$$

The equation of the plane, which touch the ellipsoid (i) is

$$lx+my+nz=p \qquad \text{................ (ii)}$$

Suppose that it touch (i) at the point (x_1, y_1, z_1). Then the equation of the tangent plane to (i) at $P(x_1, y_1, z_1)$ is -

$$\frac{x x_1}{a^2}+\frac{y y_1}{b^2}+\frac{z z_1}{c^2}=1 \qquad \text{.......... (iii)}$$

Now the equation (ii) and (iii) are the same. Hence comparing their coefficient, we get -

$$\frac{x_1}{a^2 l}+\frac{y_1}{b^2 m}+\frac{z_1}{c^2 n}=\frac{1}{p}$$

$$\therefore \quad x_1=\frac{a^2 l}{p},\ y_1=\frac{b^2 m}{p},\ z_1=\frac{c^2 n}{p}$$

Since $P(x_1, y_1, z_1)$ lines on the elliposid (i) therefore

$$\frac{x_1^{\ 2}}{a^2}+\frac{y_1^{\ 2}}{b^2}+\frac{z_1^{\ 2}}{c^2}=1$$

or, $$\frac{a^4l^2}{a^2p^2}+\frac{b^4m^2}{b^2p^2}+\frac{c^4n^2}{c^2p^2}=1$$

or, $$a^2l^2+b^2m^2+c^2n^2=p^2$$

Which is the required conditions.

Q.36 Let OP, OQ. OR be the conjugate semi-diametors of conicoid $2x^2+3y^2+4z^2=12$, then find the value of $OP^2+OQ^2+OR^2$. (SU-2017)

Solⁿ: Since $2x^2+3y^2+4z^2=12$

or, $$\frac{x^2}{6}+\frac{y^2}{4}+\frac{x^2}{3}=1$$

$\because$ $$OP^2+OQ^2+OR^2=a^2+b^2+c^2$$

Or, $$= 6+4+3=13 \quad \text{Ans.}$$

Q.37 Let OP, OQ. OR be the conjugate semi-diametors of conicoid $2x^2+3y^2+\lambda z^2=2$, find λ if $OP^2+OQ^2+OR^2=10$. (DDU-2017)

Solⁿ: Equation of central conicoid is

$$2x^2+3y^2+\lambda z^2=2$$

or, $$\frac{x^2}{1}+\frac{y^2}{2/3}+\frac{x^2}{2/\lambda}=1$$

$\because$ $$OP^2+OQ^2+OR^2=a^2+b^2+c^2$$

$\therefore$ $$10=1+\frac{2}{3}+\frac{2}{\lambda}$$

or, $$10-1-\frac{2}{3}=\frac{2}{\lambda}\Rightarrow 9-\frac{2}{3}=\frac{2}{\lambda}\Rightarrow\frac{2}{\lambda}=\frac{25}{3}$$

or, $$\lambda=\frac{25}{6}$$

Q. 38 : Prove that the centre of the sectionof the ellipsoid $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$ by the plane ABC whose equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ is the centroid of the triangle ABC. (DDU-2017)

Solⁿ : The equation of the elliposid is

$$S\equiv\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}-1=0 \quad \text{........ (1)}$$

and the equation of the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{........ (2)}$$

Let $\alpha,\beta,\gamma)$ be the centre of the section of (1) by the plane (2) then the equation of this section is $T=S_1$.

i.e. $$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}-1=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1$$

or, $$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad \text{........ (3)}$$

Also as the point $\alpha,\beta,\gamma)$ lies on (2) so

$$\Sigma\frac{\alpha}{a}=1 \quad \text{....... (4)}$$

The equation (2) and (3) represent the same plane, so comparing them we get

$$\frac{\alpha/a^2}{1/a}=\frac{\beta/b^2}{1/b}=\frac{\gamma/c^2}{1/c}\Rightarrow\frac{\alpha^2/a^2}{1}=\frac{\beta^2/b^2}{1}=\frac{\gamma^2/c^2}{1}=k\,(say)$$